曲线坐标系和张量分析（I）

§ 曲线坐标系

P为 R^3 中的一点，其坐标可以用 (x,y,z) 来表示。更一般的用 (u^1,u^2,u^3) 来表示，它们是 x,y,z 的函数，即

$$u^1=u^1(x,y,z),\quad u^2=u^2(x,y,z),\quad u^3=u^3(x,y,z).$$

比如，球面坐标 (r,θ,φ)，柱坐标 (r,θ,z)。反之，$x.y.z$ 也可以表示成 u^1,u^2,u^3 的函数。我们要求 Jacobian：$\dfrac{\partial(u^1,u^2,u^3)}{\partial(x,y,z)}\neq 0$。

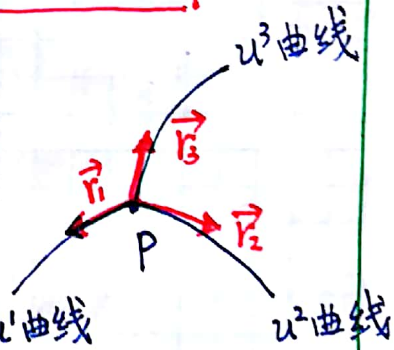
设 $\vec{r}$ 点，$\vec{r}=\vec{r}(x,y,z)=\vec{r}(u^1,u^2,u^3)$，定义切向量：

$\vec{r}_1=\dfrac{\partial\vec{r}}{\partial u_1}$，$\vec{r}_2=\dfrac{\partial\vec{r}}{\partial u_2}$，$\vec{r}_3=\dfrac{\partial\vec{r}}{\partial u_3}$，它们是 u^1,u^2,u^3 曲线在 $\vec{r}$ 点上三个切向量。

$$d\vec{r}=\vec{r}_1du_1+\vec{r}_2du_2+\vec{r}_3du_3$$

$$\Rightarrow\ \boxed{ds^2=d\vec{r}\cdot d\vec{r}=g_{ij}\,du^idu^j,\quad g_{ij}=\vec{r}_i\cdot\vec{r}_j}$$

$$g\triangleq\begin{vmatrix}\frac{\partial x}{\partial u^1}&\frac{\partial y}{\partial u^1}&\frac{\partial z}{\partial u^1}\\\frac{\partial x}{\partial u^2}&\frac{\partial y}{\partial u^2}&\frac{\partial z}{\partial u^2}\\\frac{\partial x}{\partial u^3}&\frac{\partial y}{\partial u^3}&\frac{\partial z}{\partial u^3}\end{vmatrix}\begin{vmatrix}\frac{\partial x}{\partial u^1}&\frac{\partial x}{\partial u^2}&\frac{\partial x}{\partial u^3}\\\frac{\partial y}{\partial u^1}&\frac{\partial y}{\partial u^2}&\frac{\partial y}{\partial u^3}\\\frac{\partial z}{\partial u^1}&\frac{\partial z}{\partial u^2}&\frac{\partial z}{\partial u^3}\end{vmatrix}=\begin{vmatrix}g_{11}&g_{12}&g_{13}\\g_{21}&g_{22}&g_{23}\\g_{31}&g_{32}&g_{33}\end{vmatrix}$$



体积元

$$dv=dxdydz=\frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)}du^1du^2du^3=\sqrt{g}\,du^1du^2du^3$$

例：球面坐标 $u^1=r,\ u^2=\theta,\ u^3=\varphi \Rightarrow ds^2=dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\varphi^2$

$$g_{ij}=\begin{bmatrix}1 & & \\ & r^2 & \\ & & r^2\sin^2\theta\end{bmatrix},\quad \det g=r^4\sin^2\theta.\ \Rightarrow dv=r^2\sin\theta\, dr\,d\theta\,d\varphi$$

柱面坐标 $ds^2=dr^2+r^2d\varphi^2+dz^2$

$$g_{ij}=\begin{bmatrix}1 & & \\ & r^2 & \\ & & 1\end{bmatrix},\quad \det g=r^2.\ \Rightarrow dv=r\,dr\,d\theta\,d\varphi$$

2：逆变向量和协变向量

考虑一个线性空间 V，有基向量 $\{v_i\}$。基向量的变换矩阵：

$\bar{v}_i = v_j a^j_i$，$\{\bar{v}_i\}$ 是另一组新的基。

定义 $A=\begin{bmatrix}a^1_1 & \cdots & a^1_n\\ a^2_1 & \cdots & a^2_n\\ \vdots & & \\ a^n_1 & \cdots & a^n_n\end{bmatrix} \Rightarrow \boxed{\{\bar{v}_1,\bar{v}_2,\cdots\bar{v}_n\}=\{v_1,\cdots v_2\}A}$.

★ V 中的向量 v 称为逆变向量 $v=c^iv_i=\bar{c}^i\bar{v}_i$

$$v=\{v_1\cdots v_n\}\begin{pmatrix}c^1\\c^2\\ \vdots\\ c^n\end{pmatrix}=\{\bar{v}_1,\cdots\bar{v}_n\}\begin{pmatrix}\bar{c}^1\\ \vdots\\ \bar{c}^n\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}c^1\\ \vdots\\ c^n\end{pmatrix}=A\begin{pmatrix}\bar{c}^1\\ \vdots\\ \bar{c}^n\end{pmatrix} \Rightarrow \boxed{\begin{pmatrix}\bar{c}^1\\ \vdots\\ \bar{c}^n\end{pmatrix}=A^{-1}\begin{pmatrix}c^1\\ \vdots\\ c^n\end{pmatrix}}$$

$$\boxed{a^i_j b^j_k=\delta^i_k \quad c^i=a^i_j\bar{c}^j \Rightarrow \bar{c}^i=b^i_j c^j}$$

所以，当 v 不变，其分量的变换规律是和基矢的变换相逆的，所以 v 称为逆变矢量，(contravariant vector)，而 V 被称为逆变矢量空间。

☆ 协变矢量是在 V 的对偶空间里的矢量，即 V 上所有线性映射所组成的空间。设 $\{\omega^i\}$ 是 V' 中与 $\{v_i\}$ 对偶的基，有

$$\omega^i\{v_j\} = \delta^i_j \ , \quad i,j = 1,2,\cdots n.$$

设 $\bar{v}_i = v_j a^j_i$, and $\{\bar{\omega}^i\}$ 是与 $\{\bar{v}_i\}$ 对偶的一组基，则

$$\bar{\omega}^i\{\bar{v}_j\} = \delta^i_j \quad \Rightarrow \quad \begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \vdots \\ \bar{\omega}^n \end{pmatrix} (\bar{v}_1 \cdots \bar{v}_n) = I$$

$$\Rightarrow \quad A^{-1} \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} \underbrace{(v_1 \cdots v_n)\, A}_{(\bar{v}_1 \cdots \bar{v}_n)} = A^{-1} A = I \Rightarrow \boxed{\begin{pmatrix} \bar{\omega}^1 \\ \vdots \\ \bar{\omega}^n \end{pmatrix} = A^{-1} \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix}}$$

即 $\boxed{\bar{\omega}^i = b^i_j\, \omega^j}$

设 $\omega \in V'$, $\quad \omega = (s_1 \cdots s_n) \begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} = (\bar{s}_1 \cdots \bar{s}_n) \begin{pmatrix} \bar{\omega}^1 \\ \vdots \\ \bar{\omega}^n \end{pmatrix}$

$$\Rightarrow (s_1 \cdots s_n) = (\bar{s}_1 \cdots \bar{s}_n)\, A^{-1} \qquad \Rightarrow (\bar{s}_1 \cdots \bar{s}_n) = (s_1 \cdots s_n)\, A$$

$\boxed{\bar{s}_i = a^j_i\, s_j}$ ← 对偶空间的矢量的分量变换形式和其原线性空间基矢变换一样，故称协变矢量。

$$\boxed{\begin{pmatrix} \bar{s}_1 \\ \vdots \\ \bar{s}_n \end{pmatrix} = A^t \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}}$$

回到坐标，x^i 为逆变，其变换矩阵记为 M， ← 就是原来的记号 A^{-1}

$$\boxed{\bar{x}^i = M^i_j\, x^j .}$$

记 x_i 为协变分量 $\quad \boxed{\bar{x}_i = M^{-1\,j}_{\ \ i}\, x_j \quad \text{or} \quad x_i = M^j_i\, \bar{x}_j}$

Scanned by CamScanner

$$M^i_j = \frac{\partial \bar{x}^i}{\partial x^j}$$

$$M = \frac{\partial(\bar{x}^1 \cdots \bar{x}^n)}{\partial(x^1 \cdots x^n)} = \begin{bmatrix} \frac{\partial \bar{x}^1}{\partial x^1}, & \cdots & \frac{\partial \bar{x}^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial \bar{x}^n}{\partial x^1}, & \cdots & \frac{\partial \bar{x}^n}{\partial x^n} \end{bmatrix}$$

$$M^{-1} = \frac{\partial(x^1 \cdots x^n)}{\partial(\bar{x}^1, \cdots \bar{x}^n)} = \begin{bmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \cdots & \frac{\partial x^1}{\partial \bar{x}^n} \\ \vdots & & \vdots \\ \frac{\partial x^n}{\partial \bar{x}^1} & \cdots & \frac{\partial x^n}{\partial \bar{x}^n} \end{bmatrix}$$

定义: 逆变矢量 $\bar{t}^i = \frac{\partial \bar{x}^i}{\partial x^j} t^j$

协变矢量 $\bar{t}_i = (M^{-1})^j_i \, t_j = \frac{\partial x^j}{\partial \bar{x}^i} t_j$

(p,q)型 张量分量的变换

$$\bar{t}^{i_1 \cdots i_p}_{j_1 \cdots j_q} = \frac{\partial \bar{x}^{i_1}}{\partial x^{l_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{l_p}} \cdot \frac{\partial x^{k_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{k_q}}{\partial \bar{x}^{j_q}} \, t^{l_1 \cdots l_p}_{k_1 \cdots k_q}$$

微分 $d\bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial x^\nu} dx^\nu$, $\frac{\partial}{\partial \bar{x}^\mu} = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \frac{\partial}{\partial x^\nu}$ ← 协变张量

§ 曲线坐标下的张量

对于曲线坐标 $\underbrace{}_{(u^1,u^2,u^3)}$，我们有自然标架 $\vec{r}_i = \frac{\partial \vec{r}}{\partial u_i}$. 设对另一套曲线坐标 $(\bar{u}^1, \bar{u}^2, \bar{u}^3)$, 我们有自然标架 ($\vec{\bar{r}}_i = \frac{\partial \vec{r}}{\partial \bar{u}_i}$). 设在 P点, 的向量

$$\vec{v} = v^i \vec{r}_i = \bar{v}^i \vec{\bar{r}}_i .$$

$$\vec{\bar{r}}_i = \frac{\partial u^j}{\partial \bar{u}^i} \frac{\partial \vec{r}}{\partial u^j} \Rightarrow v^j \vec{r}_j = \bar{v}^i \frac{\partial u^j}{\partial \bar{u}^i} \vec{r}_j$$

$$\Rightarrow \boxed{v^j = \frac{\partial u^j}{\partial \bar{u}^i} \bar{v}^i \quad \text{or} \quad \bar{v}^i = \frac{\partial \bar{u}^i}{\partial u^j} v^j \quad \text{所以 } v^i \text{ 是逆变分量}}$$

向量的协变分量：

$$v_i = \vec{v}\cdot\vec{r}_i = v^j \vec{r}_j\cdot\vec{r}_i = v^j g_{ji}$$

其中 $g_{ij} = \vec{r}_i\cdot\vec{r}_j$ 是一个2阶协变张量，叫 metric（度规）张量。

$$\boxed{\bar{g}_{ij} = \vec{\bar{r}}_i\cdot\vec{\bar{r}}_j = \frac{\partial u^{i'}}{\partial \bar{u}^i}\frac{\partial u^{j'}}{\partial \bar{u}^j} g_{i'j'}} \qquad (*)$$

$$\bar{v}_i = \vec{v}\cdot\vec{\bar{r}}_i = \vec{v}\cdot\frac{\partial u^j}{\partial \bar{u}^i}\vec{r}_j = \frac{\partial u^j}{\partial \bar{u}^i} v_j$$ ← 所以说 v_i 是协变分量。

由 $(*)$ $\Rightarrow$ $\bar{g} = J^2 g$, where $J = \det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right)$.

$\bar{g} = \det \bar{g}_{ij}$, $g = \det g_{ij}$

$$\Rightarrow \boxed{\sqrt{\bar{g}} = \sqrt{g}\, J}$$ ← $\sqrt{g}$ 是权为1的标量，权即Jocabin的幂次。

下面定义度规张量的逆变形式。为方便记，用 g 代表矩阵形式，协变度规

G 代表逆变度规形式

$\mathcal{U}$ 代表 $\frac{\partial u^i}{\partial \bar{u}^j}$

$\mathcal{U}^{-1}$ 代表 $\frac{\partial \bar{u}^i}{\partial u^j}$

由 $v_i = v^j g_{ji} \Rightarrow v^j = (g^T)^{-1}_{ji} v_i$

即 $G = (g^T)^{-1}$.

在坐标变换下，$\bar{g} = \mathcal{U}^T g\, \mathcal{U}$

$$\Rightarrow \bar{G} = (\bar{g}^T)^{-1} = (\mathcal{U}^T g^T \mathcal{U})^{-1} = \mathcal{U}^{-1}(g^T)^{-1}(\mathcal{U}^{-1})^T$$

$$= \mathcal{U}^{-1} G (\mathcal{U}^{-1})^T$$

or $\bar{G}^{ij} = \frac{\partial \bar{u}^i}{\partial u^{i'}} G^{i'j'} \frac{\partial \bar{u}^j}{\partial u^{j'}}$ $\Rightarrow$ G 是逆变张量.

一般我们还是用 g^{ij} 代表逆变度规，即

$$\boxed{\bar{g}^{ij} = \frac{\partial \bar{u}^i}{\partial u^{i'}}\frac{\partial \bar{u}^j}{\partial u^{j'}} g^{i'j'}}$$

$G = (g^T)^{-1}$

$$\boxed{g^{il} g_{jl} = \delta^i_j}$$

§ Christoffel symbol

在曲线坐标系下，在点 $\vec{r}$ 处的 $\vec{r}_i = \frac{\partial \vec{r}}{\partial u_i}$ 构成了 $\vec{r}$ 处的一个标架。各点处的标架是不同的，我们来研究标架的变化，把 $\partial_j \vec{r}_i$ 用 $\{\vec{r}_i\}$ 展开

$$\boxed{\partial_j \vec{r}_i = \frac{\partial^2 \vec{r}}{\partial u^j \partial u^i} = \Gamma^k_{ji} \vec{r}_k = \Gamma^k_{ji} \frac{\partial \vec{r}}{\partial u^k}}$$

i) $\Gamma^k_{ji} = \Gamma^k_{ij}$，因为 $\partial_j \vec{r}_i = \partial_i \vec{r}_j = \frac{\partial^2 \vec{r}}{\partial u^i \partial u^j}$

2) $$\boxed{\Gamma^k_{ji} = \frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji})}$$

证：$g_{il} = \vec{r}_i \cdot \vec{r}_l \Rightarrow \partial_j g_{il} = (\partial_j \vec{r}_i) \cdot \vec{r}_l + \vec{r}_i \cdot (\partial_j \vec{r}_l)$

$$\partial_j g_{il} = \Gamma^h_{ji} \vec{r}_h \cdot \vec{r}_l + \vec{r}_i \cdot \Gamma^h_{jl} \vec{r}_h = \Gamma^h_{ji} g_{hl} + \Gamma^h_{jl} g_{ih} \quad ①$$

$$\to \partial_i g_{jl} = \Gamma^h_{ij} g_{hl} + \Gamma^h_{il} g_{jh} \quad ②$$

$$\to \partial_l g_{ji} = \Gamma^h_{lj} g_{hi} + \Gamma^h_{li} g_{jh} \quad ③$$

① + ② − ③ $\quad \partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji} = 2 \Gamma^h_{ji} g_{hl}$

$$\frac{1}{2} g^{kl} (\partial_j g_{il} + \partial_i g_{jl} - \partial_l g_{ji}) = \Gamma^h_{ji} \underbrace{g_{hl} g^{kl}}_{\delta^k_h} = \Gamma^k_{ji} \quad ✓.$$

例 $\Gamma^j_{kj} = \partial_k \ln \sqrt{g}$

Proof: $g = \det g_{ij}$ and $\ln \det g_{ij} = \mathrm{tr} \ln g_{ij} = \ln g$

$$\partial_k \ln g = \mathrm{tr}[\, g^{-1} \partial_k g \,] = {g^{-1}}_{ji} \partial_k g_{ij} = g^{ij} \partial_k g_{ij}$$

用了 $\boxed{\ln \det A = \mathrm{tr} \ln A.}$ 这里 g 解释为矩阵 $\quad g^{ij} = (g^{T-1})_{ij}$

$$\Gamma^j_{kj} = \frac{1}{2} g^{jl} (\partial_k g_{jl} + \underbrace{\partial_j g_{kl} - \partial_l g_{kj}}_{jl\text{ 对称}}) = \frac{1}{2} g^{jl} \partial_k g_{jl} = \frac{1}{2} \partial_k \ln g$$

$$= \partial_k \ln \sqrt{g}.$$

3) 在坐标变换 $u^i \to \bar{u}^i$ 下，有

$$\frac{\partial u^k}{\partial \bar{u}^{k'}} \bar{\Gamma}^{k'}_{j'i'} = \frac{\partial u^j}{\partial \bar{u}^{j'}} \frac{\partial u^i}{\partial \bar{u}^{i'}} \Gamma^k_{ji} + \frac{\partial^2 u^k}{\partial \bar{u}^{i'} \partial \bar{u}^{j'}}$$

Proof: 在 $\bar{u}^{i'}$ 坐标系下，$\partial_{j'} \vec{r}_{i'} = \bar{\Gamma}^{k'}_{j'i'} \vec{r}_{k'}$, where $\vec{r}_{i'} = \frac{\partial \vec{r}}{\partial \bar{u}^{i'}}$

$$\partial_{j'} \vec{r}_{i'} = \partial_{j'} \left[\frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial \vec{r}}{\partial u^i} \right] = \frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial \vec{r}}{\partial u^i}$$

$$= \frac{\partial^2 u^i}{\partial \bar{u}^{i'} \partial \bar{u}^{j'}} \frac{\partial \vec{r}}{\partial u^i} + \frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial}{\partial \bar{u}^{j'}} \left(\frac{\partial \vec{r}}{\partial u^i} \right)$$

$$= \frac{\partial^2 u^k}{\partial \bar{u}^{i'} \partial \bar{u}^{j'}} \frac{\partial \vec{r}}{\partial u^k} + \frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial u^j}{\partial \bar{u}^{j'}} \frac{\partial^2 \vec{r}}{\partial u^i \partial u^j} \leftarrow \Gamma^k_{ij} \frac{\partial \vec{r}}{\partial u^k}$$

$$= \left[\frac{\partial^2 u^k}{\partial \bar{u}^{i'} \partial \bar{u}^{j'}} + \frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial u^j}{\partial \bar{u}^{j'}} \Gamma^k_{ij} \right] \frac{\partial \vec{r}}{\partial u^k}$$

另一方面 $\partial_{j'} \vec{r}_{i'} = \bar{\Gamma}^{k'}_{j'i'} \vec{r}_{k'} = \bar{\Gamma}^{k'}_{j'i'} \frac{\partial u^k}{\partial \bar{u}^{k'}} \vec{r}_k \leftarrow \frac{\partial \vec{r}}{\partial u^k}$

$$\Rightarrow \bar{\Gamma}^{k'}_{j'i'} \frac{\partial u^k}{\partial \bar{u}^{k'}} = \frac{\partial^2 u^k}{\partial \bar{u}^{i'} \partial \bar{u}^{j'}} + \frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial u^j}{\partial \bar{u}^{j'}} \Gamma^k_{ij}$$

$$\boxed{\bar{\Gamma}^{k'}_{j'i'} = \frac{\partial \bar{u}^{k'}}{\partial u^k} \frac{\partial^2 u^k}{\partial \bar{u}^{i'} \partial \bar{u}^{j'}} + \frac{\partial \bar{u}^{k'}}{\partial u^k} \frac{\partial u^i}{\partial \bar{u}^{i'}} \frac{\partial u^j}{\partial \bar{u}^{j'}} \Gamma^k_{ij}} \leftarrow \Gamma^k_{ij} \text{ 不是张量.}$$

4) $\partial_j \Gamma^l_{ki} - \partial_k \Gamma^l_{ji} + \Gamma^l_{jh} \Gamma^h_{ki} - \Gamma^l_{kh} \Gamma^h_{ji} = 0$.

Proof: $\partial_j \partial_k \frac{\partial \vec{r}}{\partial u^i} = \partial_k \partial_j \frac{\partial \vec{r}}{\partial u^i} \Rightarrow \partial_j (\Gamma^l_{ki} \vec{r}_l) = \partial_k [\Gamma^l_{ji} \vec{r}_l]$

$$\partial_j \Gamma^l_{ki} \vec{r}_l + \Gamma^l_{ki} \partial_j \vec{r}_l = \partial_k \Gamma^l_{ji} \vec{r}_l + \Gamma^l_{ji} \partial_k \vec{r}_l$$

$$\Gamma^l_{ki}\partial_j\vec{r}_l = \Gamma^h_{ki}\partial_j\vec{r}_h = \Gamma^h_{ki}\Gamma^l_{jh}\vec{r}_l$$

$$\Gamma^l_{ji}\partial_k\vec{r}_l = \Gamma^h_{ji}\partial_k\vec{r}_h = \Gamma^h_{ji}\Gamma^l_{kh}\vec{r}_l$$

$$\Rightarrow \boxed{\partial_j\Gamma^l_{ki} - \partial_k\Gamma^l_{ji} + \Gamma^l_{jh}\Gamma^h_{ki} - \Gamma^l_{kh}\Gamma^h_{ji} = 0.}$$

这是由定义 Christoffel symbol 时用了三维平直空间

定义 K^l_{jki} $\Rightarrow$ R^3 平直空间的曲率张量 $K^l_{jki} = 0$。

§ 协变微分

$\vec{v} = v^i\vec{r}_i$ 用 $\vec{r}_i$ 做基来展开矢量 $\vec{v}$，$\vec{r}_i = \frac{\partial\vec{r}}{\partial u^i}$

$$d\vec{v} = dv^i\vec{r}_i + v^j\frac{\partial\vec{r}_j}{\partial u^k}du^k = (dv^i + v^j\Gamma^i_{kj}du^k)\vec{r}_i$$

$$= (\partial_k v^i + v^j\Gamma^i_{kj})du^k\,\vec{r}_i$$

记 $d\vec{v} = \delta v^i\vec{r}_i$ $\Rightarrow$ $\delta v^i = (\partial_k v^i + v^j\Gamma^i_{kj})du^k$

$\boxed{\delta v^i \text{ 叫协变微分}}$ $\quad \delta v^i \equiv D_k v^i du^k$

则 $\boxed{D_k v^i = \partial_k v^i + \Gamma^i_{kj}v^j \quad \text{对 } v^i \text{ 的协变导数.}}$

下面推导协变分量的协变微分

$$\bar{v}_i\bar{v}^i = \frac{\partial u^j}{\partial\bar{u}^i}v_j\cdot\frac{\partial\bar{u}^i}{\partial u^{j'}}v^{j'} = \delta^j_{j'}v_jv^{j'} = v_jv^j$$

Hence v_iv^i 是一个标量，其协变微分就是普通的微分

$$\delta(v_iv^i) = d(v_iv^i) \Rightarrow \delta v_i v^i + v_i\delta(v^i) = dv_i v^i + v_i dv^i$$

(我们要求协变微分满足 Leibniz 法则)

$$\delta v_i v^i + v_i(\cancel{\partial_k v^i} + \Gamma^i_{ki}v^j)du^k = dv_i v^i + \cancel{v_i dv^i}$$

$$\delta v_i v^i = dv_i v^i - v_j \Gamma^j_{ki} v^i du^k \Rightarrow \delta v_i = dv_i - \Gamma^k_{ji} v_k du^j$$

$$dv_i = \partial_j v_i du^j$$

$$\delta v_j = D_j v_i du^j$$

$$\Rightarrow \boxed{D_j v_i = \partial_j v_i - \Gamma^k_{ji} v_k}$$

or $\boxed{D_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k , \quad D_i v_j = \partial_i v_j - \Gamma^k_{ij} v_k}$

Similarly, we can define covariant derivative for tensors.

For example

$$\delta T^k_{ji} = dT^k_{ji} + \Gamma^k_{hl} T^l_{ji} \delta u^h - \Gamma^l_{hj} T^k_{li} \delta u^h - \Gamma^l_{hi} T^k_{jl} du^h$$

$$\rightarrow D_h T^k_{ji} = \partial_h T^k_{ji} + \Gamma^{\boxed{k}}_{hl} T^l_{ji} - \Gamma^l_{h\boxed{j}} T^k_{li} - \Gamma^l_{h\boxed{i}} T^k_{jl}$$

对k微分　对j微分　对i微分.

§ 梯度、散度、旋度作为协变导数

① 梯度：考虑一个标量场 f，在曲线坐标 $\{u^i\}$ 中表示为 $f(u^1,u^2,u^3)$，则用其协变导数定义一个向量场，其协变分量为 $D_i f = \partial_i f = v_i$。

则其逆变分量 $v^i = g^{ij} v_j \Rightarrow D^i f = g^{ij}\partial_j f$，其矢量为

$$\boxed{D^i f \cdot \vec{r}_i = g^{ij}\partial_j f \cdot \frac{\partial \vec{r}}{\partial u^i} \triangleq \vec{\nabla} f}$$

梯度定义与坐标无关：在另一组坐标 $\{\bar{u}^i\}$ 下，有

向量

$$\bar{g}^{ij} = \frac{\partial \bar{u}^i}{\partial u^{i'}}\frac{\partial \bar{u}^j}{\partial u^{j'}} g^{i'j'}, \quad \partial_{\bar{u}^j} f = \frac{\partial u^{j'}}{\partial \bar{u}^{j}}\partial_{u^{j'}} f$$

$$\frac{\partial \vec{r}}{\partial \bar{u}^i} = \frac{\partial u^{i'}}{\partial \bar{u}^i}\frac{\partial \vec{r}}{\partial u^{i'}}$$

$$\Rightarrow \bar{g}^{ij}\partial_{\bar{u}^j} f\cdot\frac{\partial \vec{r}}{\partial \bar{u}^i} = g^{i'j'}\frac{\partial \bar{u}^i}{\partial u^{i'}}\underbrace{\frac{\partial \bar{u}^j}{\partial u^{j'}}\frac{\partial u^{j''}}{\partial \bar{u}^j}}_{\delta^{j''}_{j'}}\frac{\partial u^{i''}}{\partial \bar{u}^i}\partial_{u^{j''}} f\frac{\partial \vec{r}}{\partial u^{i''}}$$

$$\delta^{i''}_{i'}$$

$$= g^{i'j'}\partial_{j'} f\frac{\partial \vec{r}}{\partial u^{i'}}$$

$$= g^{ij}\frac{\partial f}{\partial u^j}\frac{\partial \vec{r}}{\partial u^i}$$

→ 在直角坐标系下 $\vec{\nabla} f = \partial_x f\,\hat{x} + \partial_y f\,\hat{y} + \partial_z f\,\hat{z}$.

在正交曲线坐标系 $g_{ii} = \mathrm{diag}\left[\frac{\partial \vec{r}}{\partial u_i}\cdot\frac{\partial \vec{r}}{\partial u_i}\right]$，$g^{ii} = \left(\frac{\partial \vec{r}}{\partial u_i}\cdot\frac{\partial \vec{r}}{\partial u_i}\right)^{-1}$

u^i 方向单位矢量 $\hat{u}^i = \frac{1}{\sqrt{g_{ii}}}\frac{\partial \vec{r}}{\partial u^i}$

$$\Rightarrow \boxed{\nabla f = g^{ii}\frac{\partial f}{\partial u^i}\sqrt{g_{ii}}\,\hat{u}^i = \underbrace{\frac{1}{\sqrt{g_{ii}}}\frac{\partial f}{\partial u^i}}_{\text{定义为: } w^i}\hat{u}_i}$$

柱坐标 $\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi + \frac{\partial f}{\partial z}\hat{e}_z$

比如：球坐标 $\nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi$

② 散度：对于矢量场 $\vec{v} = v^i \vec{r}_i$，其协变导数

$$D_j v^i = \partial_j v^i + \Gamma^i_{jk} v^k \xrightarrow{\text{缩并}} D_j v^j = \partial_j v^j + \Gamma^j_{jk} v^k$$

$$\boxed{D_j v^j = \partial_j v^j + \frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial u^k} v^k = \frac{1}{\sqrt{g}}\frac{\partial(\sqrt{g}\,u^k)}{\partial u^k}}$$

对共交曲线系

$$\boxed{\nabla\cdot\vec{v} = \frac{1}{\sqrt{g_{11}g_{22}g_{33}}}\left[\frac{\partial\sqrt{g_{22}g_{33}}\,\omega^1}{\partial u^1} + \frac{\partial(\sqrt{g_{11}g_{33}}\,\omega^2)}{\partial u^2} + \frac{\partial(\sqrt{g_{22}g_{11}}\,\omega^3)}{\partial u^3}\right]}$$

比如球坐标下

$$\nabla^2 f = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(r\sin\theta\,\overset{1/r}{\cancel{}}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(r\,\frac{1}{\sin\theta\, r}\frac{\partial f}{\partial \varphi}\right)\right]$$

$$\boxed{\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial^2 \varphi}}$$

柱坐标

$$\nabla^2 f = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial f}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial f}{\partial z}\right)\right]$$

$$\boxed{\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}f + \frac{\partial^2}{\partial z^2}f}$$

$$\nabla^2 f \equiv D_i D^i f = D_i(g^{ik} D_k f) = g^{ik} D_i D_k f$$

（用了 $D_k g_{ij} = D_k g^{ij} = 0$）← 可以证明（不容易）

③旋度：

定义张量 $e^{ijk} = \frac{1}{\sqrt{g}}\epsilon^{ijk}$，$e_{ijk} = \sqrt{g}\,\epsilon_{ijk}$. e^{ijk}是一个逆变张量，而 e_{ijk}是一个顺变张量。

Prof: $\bar{e}^{ijk} = \frac{1}{\sqrt{\bar{g}}}\epsilon^{ijk}$, since $\sqrt{g}\,du^1du^2du^3$ invariant

$$= \sqrt{g}\,\det\left(\frac{\partial u^i}{\partial \bar{u}^j}\right) d\bar{u}_1 \cdot d\bar{u}_3$$

$$\sqrt{\bar{g}} = J\sqrt{g} \Rightarrow \frac{1}{\sqrt{\bar{g}}} = \frac{1}{\sqrt{g}}J^{-1}$$

$$\Rightarrow \bar{e}^{ijk} = \frac{1}{\sqrt{g}}J^{-1}\epsilon^{ijk} = \frac{1}{\sqrt{g}}\frac{\partial \bar{u}^i}{\partial u^{i'}}\frac{\partial \bar{u}^j}{\partial u^{j'}}\frac{\partial \bar{u}^k}{\partial u^{k'}}\epsilon^{i'j'k'}$$

$$= \frac{\partial \bar{u}^i}{\partial u^{i'}}\frac{\partial \bar{u}^j}{\partial u^{j'}}\frac{\partial \bar{u}^k}{\partial u^{k'}} e^{i'j'k'}$$

定义旋度：对于矢量协变分量 $\vec{v} = v_i g^{ij}\vec{r}_j$，定义协变导数

$$D_j v_i = \partial_j v_i - \Gamma^k_{ji} v_k \Rightarrow D_j v_i - D_i v_j = \partial_j v_i - \partial_i v_j$$

定义 $A^h = \frac{1}{2}e^{hji}(D_j v_i - D_i v_j) = \frac{1}{2}e^{hji}(\partial_j v_i - \partial_j v_i)$ 是一个逆变矢量分量，

$$\boxed{\vec{A} = \nabla\times\vec{v} = A^h\,\vec{r}_h}$$ → 在直角坐标下，回到原始定义.

如果定义 $\vec{v} = \omega_1 \underbrace{\frac{1}{\sqrt{g_{11}}}\vec{r}_1} + \omega_2 \underbrace{\frac{1}{\sqrt{g_{22}}}\vec{r}_2} + \omega_3 \underbrace{\frac{1}{\sqrt{g_{33}}}\vec{r}_3}$

归一化单位矢量

$v_1 = g_{11}v^1 = \sqrt{g_{11}}\,\omega_1$, $v_2 = g_{22}v^2 = \sqrt{g_{22}}\,\omega_2$, $v_3 = g_{33}v^3 = \sqrt{g_{33}}\,\omega_3$

$$\Rightarrow \nabla\times\vec{v} = \frac{1}{\sqrt{g}}\Big[\big(\partial_2(\omega_3\sqrt{g_{33}}) - \partial_3(\omega_2\sqrt{g_{22}})\big]\sqrt{g_{11}}\,\frac{1}{\sqrt{g_{11}}}\vec{r}_1$$
$$+\big(\partial_3(\omega_1\sqrt{g_{11}}) - \partial_1(\omega_3\sqrt{g_{33}})\big]\sqrt{g_{22}}\,\frac{1}{\sqrt{g_{22}}}\vec{r}_2$$
$$+\big(\partial_1(\omega_2\sqrt{g_{22}}) - \partial_2(\omega_1\sqrt{g_{11}})\big]\sqrt{g_{33}}\,\frac{1}{\sqrt{g_{22}}}\vec{r}_3\Big]$$

① 对于球坐标、 $\omega_1 = v_r$, $\omega_2 = v_\theta$, $\omega_3 = v_\varphi$

$g_{11} = 1$ $\quad g_{22} = r^2$, $g_{33} = r^2\sin^2\theta \Rightarrow \sqrt{g} = r^2\sin\theta$

$$\nabla\times\vec{v} = \frac{1}{r^2\sin\theta}\Big\{[\partial_\theta(v_\varphi r\sin\theta) - \partial_\varphi(v_\theta r)]\ \hat{e}_r$$

$$+ [\partial_\varphi(v_r) - \partial_r(v_\varphi r\sin\theta)]\, r\ \hat{e}_\theta$$

$$+ [\partial_r(v_\theta r) - \partial_\theta(v_r)]\, r\sin\theta\ \hat{e}_\varphi\Big]$$

$$\boxed{\begin{aligned}\nabla\times\vec{v} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\varphi) - \frac{\partial v_\theta}{\partial\varphi}\right]\hat{e}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\varphi} - \frac{\partial}{\partial r}(r v_\varphi)\right]\hat{e}_\theta \\ &+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{e}_\varphi\end{aligned}}$$

② 对于柱坐标 $\omega_1 = v_r$, $\omega_2 = v_\varphi$ $\quad \omega_3 = v_z$

$g_{11} = 1$ $\quad g_{22} = r^2$ $\quad g_{33} = 1 \Rightarrow \sqrt{g} = r$

$$\nabla\times\vec{v} = \frac{1}{r}[\partial_\varphi(v_z) - \partial_z(v_\varphi\cdot r)]\ \hat{e}_r$$

$$+ \frac{1}{r}[\partial_z[v_r] - \partial_r[v_z]]\cdot r\ \hat{e}_\varphi$$

$$+ \frac{1}{r}[\partial_r[v_\varphi r] - \partial_\varphi[v_r]]\cdot\hat{e}_z$$

$$\boxed{\begin{aligned}\nabla\times\vec{v} &= \left(\frac{1}{r}\frac{\partial v_z}{\partial\varphi} - \frac{\partial v_\varphi}{\partial z}\right)\hat{e}_r + \left(\frac{\partial}{\partial z}v_r - \frac{\partial}{\partial r}v_z\right)\hat{e}_\varphi \\ &+ \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_\varphi) - \frac{\partial v_r}{\partial\theta}\right)\hat{e}_z\end{aligned}}$$