More exercises on covariant derivatives

. Let us rethink the curved space, but we will ~~start~~ with the curve coordinate system for a flat space, and then put constraints to arrive a curved 2D surface. In other words, we can embed the 2D curved surface into a flat 3D space, and then do the projection. Nontrivial curvature arises from projections.

1. Consider the spherical coordinate in 3D, $\vec{r} = r\hat{e}_r$

$$d\vec{r} = \hat{e}_r\, dr + r\, d\hat{e}_r = \hat{e}_r\, dr + r\, d\theta\, \hat{e}_\theta + r\sin\theta\, d\varphi\, \hat{e}_\varphi$$

$$\vec{r}_r = \hat{e}_r, \quad \vec{r}_\theta = r\hat{e}_\theta, \quad \vec{r}_\varphi = r\sin\theta\, \hat{e}_\varphi$$

$$\vec{A} = A^r \hat{e}_r + A^\theta \vec{r}_\theta + A^\varphi \vec{r}_\varphi,$$

$(A^r, A^\theta, A^\varphi)$ transfs like a vector, but they are different from the usual definition.

$$d\vec{A} = dA^r\, \hat{e}_r + dA^\theta\, \vec{r}_\theta + dA^\varphi\, \vec{r}_\varphi + A^r\, d\hat{e}_r + A^\theta\, d\vec{r}_\theta + A^\varphi\, d\vec{r}_\varphi$$

$$d\hat{e}_r = d\theta\, \hat{e}_\theta + \sin\theta\, d\varphi\, \hat{e}_\varphi = \frac{1}{r} d\theta\, \vec{r}_\theta + \frac{1}{r} d\varphi\, \vec{r}_\varphi$$

$$d\vec{r}_\theta = dr\, \hat{e}_\theta + r\, d\hat{e}_\theta = \frac{dr}{r} \vec{r}_\theta + r(-d\theta\, \hat{e}_r + \cos\theta\, d\varphi\, \hat{e}_\varphi)$$

$$= -r\, d\theta\, \hat{e}_r + \frac{dr}{r} \vec{r}_\theta + \cot\theta\, d\varphi\, \vec{r}_\varphi$$

$$d\vec{r}_\varphi = dr\sin\theta\, \hat{e}_\varphi + r\cos\theta\, d\theta\, \hat{e}_\varphi + r\sin\theta\, d\hat{e}_\varphi$$

$$= r\sin\theta\,(-\sin\theta\, d\varphi\, \hat{e}_r - \cos\theta\, d\varphi\, \hat{e}_\theta) + \left(\frac{dr}{r} + \cot\theta\, d\theta\right) r\sin\theta\, \hat{e}_\varphi$$

$$= -r\sin^2\theta\, d\varphi\, \hat{e}_r - \sin\theta\cos\theta\, d\varphi\, \vec{r}_\theta + \left(\frac{dr}{r} + \cot\theta\, d\theta\right) \vec{r}_\varphi$$

$$d\vec{A} = \hat{e}_r \left[dA^r - rA^\theta\, d\theta - r\sin^2\theta\, A^\varphi\, d\varphi \right]$$

$$+ \vec{r}_\theta \left[dA^\theta + \frac{A^r}{r} d\theta + \frac{A^\theta dr}{r} - A^\varphi \sin\theta\cos\theta\, d\varphi \right]$$

$$+ \vec{r}_\varphi \left[dA^\varphi + \frac{A^r}{r} d\varphi + A^\theta \cot\theta\, d\varphi + A^\varphi \left(\frac{dr}{r} + \cot\theta\, d\theta \right) \right]$$

Compare parallel transport $d\vec{A} = 0 \Rightarrow$

$$A^{\mu *}(x+dx) - A^\mu(x) = dA^\mu = -\Gamma^\mu_{\nu\lambda}\, dx^\nu A^\lambda$$

we have in a parallel ~~transport~~

$$\boxed{\begin{aligned} dA^r &= r\,d\theta\, A^\theta + r\sin^2\theta\, d\varphi\, A^\varphi \\ dA^\theta &= -\frac{1}{r} dr\, A^\theta - \frac{1}{r} d\theta\, A^r + \sin\theta\cos\theta\, d\varphi\, A^\varphi \\ dA^\varphi &= \left[-\frac{1}{r} dr\, A^\varphi - \cot\theta\, d\theta\, A^\varphi - \frac{1}{r} d\varphi\, A^r - \cot\theta\, d\varphi\, A^\theta \right] \end{aligned}}$$

Hence

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta$$

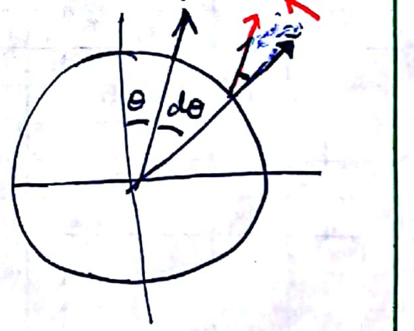
$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r, \quad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = 1/r, \quad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = \cot\theta$$

The above results look quite complicated: But it is actually the parallel transport in 3D flat space:

When $A^r \hat{e}_r$ is ~~transported~~ along a longitude by a $d\theta$ angle, we have $dA^\theta = \frac{-d\theta}{r}$,

or $dA^\theta\, r\hat{e}_\theta = -d\theta\, \hat{e}_\theta$



$$\boxed{DA^\mu = dA^\mu + \Gamma^\mu_{\nu\lambda}\, dx^\nu A^\lambda} \leftarrow \text{parallel transport.}$$

② Polar coordinate

$$ds^2 = dr^2 + r^2 d\theta^2 \quad \Rightarrow g_{rr} = 1,\ g_{\theta\theta} = r^2;\quad g^{rr} = 1,\quad g^{\theta\theta} = 1/r^2.$$

$$\partial_\mu g_{\nu\lambda} = 2r\,\delta_{\mu r}\ \delta_{\nu\lambda}\,\delta_{\lambda\theta},$$

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

$$\Rightarrow \boxed{\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r, \quad \Gamma^r_{\theta\theta} = -r}$$

or $\quad d\vec{r} = \hat{e}_r dr + r d\hat{e}_r = \hat{e}_r dr + r d\theta\, \hat{e}_\theta$

$$\vec{A} = A^r \hat{e}_r + A^\theta \vec{r}_\theta, \qquad \vec{r}_r = \hat{e}_r,\quad \vec{r}_\theta = r\hat{e}_\theta.$$

$$d\vec{A} = dA^r \hat{e}_r + dA^\theta \vec{r}_\theta + A^r d\hat{e}_r + A^\theta d\vec{r}_\theta$$

$$d\hat{e}_r = d\theta\, \hat{e}_\theta = \frac{d\theta}{r} \vec{r}_\theta,$$

$$d\vec{r}_\theta = dr\, \hat{e}_\theta + r d\hat{e}_\theta = \frac{dr}{r} \vec{r}_\theta - r d\theta\, \hat{e}_r$$

$$\Rightarrow d\vec{A} = \hat{e}_r (dA^r - \overset{A^\theta}{r} d\theta) + \vec{r}_\theta \left(dA^\theta + \frac{d\theta}{r} A^r + \frac{dr}{r} A^\theta\right)$$

$$\Rightarrow \begin{cases} dA^r - rA^\theta d\theta = 0 \\ dA^\theta + \frac{1}{r} d\theta A^r + \frac{1}{r} dr A^\theta = 0 \end{cases} \Rightarrow \begin{cases} \Gamma^r_{\theta\theta} = -r \\ \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r \end{cases}$$

More generally, consider $\vec{a}(t) = a^1(t)\,\vec{r}_1(t) + a^2(t)\,\vec{r}_2(t)$ lying in the tangent plane.

$$d\vec{a} = \sum_i da^i\,\vec{r}_i + a^i\,d\vec{r}_i = \sum_i da^i\,\vec{r}_i + a^i \sum_j \vec{r}_{ij}\,dx^j$$

$\vec{r}_{ij} = \Gamma^m_{ij}\,\vec{r}_m = \Gamma^k_{ij}\,\vec{r}_k + L_{ij}\,\vec{n}$, where $\vec{r}_k$ lies in the tangent plane, $\vec{n}$ is the norm.

$$\Rightarrow d\vec{a} = \sum_i da^i\,\vec{r}_i + a^i \sum_j \left(\Gamma^k_{ij}\,\vec{r}_k + L_{ij}\,\vec{n}\right)dx^j$$

projected out

① For $\vec{a}$, $\vec{r}_i$ only lies in the ~~tangent~~ plane, "i"-already projected.

Γ^k_{ij}

② "k"-index, — projected

③ we also need to keep dx^j along the tangent plane.

Hence

$$d\vec{a} = \sum_i \Big(da^i + \sum_{\alpha\beta} \Gamma^i_{\alpha\beta}\,a^\alpha dx^\beta\Big)\vec{r}_i + \cdots$$

$$\boxed{D\vec{a} = \sum_i \Big(da^i + \sum_{\alpha\beta} \Gamma^i_{\alpha\beta}\,a^\alpha dx^\beta\Big)\vec{r}_i}$$

After projection to the surface, we acquire curvature.

Example of application to Lagrange equation:

$$L = \frac{1}{2} g_{ij}(x)\, \dot{x}^i \dot{x}^j - V(x_i)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \Rightarrow \frac{d}{dt}\left(g_{ij}\dot{x}^j\right) - \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\dot{x}^k\dot{x}^j + \frac{\partial V}{\partial x^i} = 0$$

$$g_{ij}\ddot{x}^j + \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j - \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\dot{x}^k\dot{x}^j = -\frac{\partial V}{\partial x^i}$$

$$g_{ij}\ddot{x}^j + \left(\partial_k g_{ij} - \tfrac{1}{2}\partial_i g_{jk}\right)\dot{x}^j\dot{x}^k = -\partial_i V = F_i$$

$$g_{ij}\ddot{x}^j + \frac{1}{2}\left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right)\dot{x}^j\dot{x}^k = F_i$$

$$\ddot{x}^i + \frac{1}{2} g^{il}\left(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{jk}\right)\dot{x}^j\dot{x}^k = F_i$$

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = F_i$$, define $u^i = \frac{dx^i}{dt}$

$$\Rightarrow \boxed{\frac{du^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}u^k = 0}$$

← if $F_i = 0$, it's nothing but the parallel transport of the velocity u^i.

This is also similar to the equation of geodesic lines on a curved surface

$$l = \int_{t_0}^{t_1} dt \sqrt{g_{ij}(x)\,\dot{x}_i\dot{x}_j} = \int ds$$

variational principle

Set $T = \frac{ds}{dt} = \sqrt{g_{ij}\dot{x}^i\dot{x}^j}$ $\Rightarrow$ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^i}\right) - \frac{\partial T}{\partial x^i} = 0$

$$\frac{\partial T}{\partial \dot{x}^i} = \frac{1}{T} g_{ij}\dot{x}^j \qquad \frac{\partial T}{\partial x^i} = \frac{1}{2T}\left(\partial_i g_{jk}\right)\dot{x}^j\dot{x}^k$$

$$\frac{d}{dt}\left(\frac{1}{T} g_{ij}\dot{x}^j\right) = \frac{-1}{T^2}\frac{dT}{dt} g_{ij}\dot{x}^j + \frac{1}{T} g_{ij}\ddot{x}^j + \cancel{\frac{1}{T}\partial_i g_{kj}\dot{x}^k\dot{x}^j}$$

$$+ \frac{1}{T}\partial_k g_{ij}\,\dot{x}^j\dot{x}^k$$

$$\Rightarrow \quad g_{ij}\ddot{x}^j + \left(\partial_k g_{ij} - \frac{1}{2}\partial_i g_{jk}\right)\dot{x}^j\dot{x}^k - \frac{1}{T}\frac{dT}{dt} g_{ij}\dot{x}^j = 0$$

g^{li} multiplied to both side

$$\Rightarrow$$

$$\ddot{x}^l + \frac{1}{2} g^{li}\left(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}\right)\dot{x}^j\dot{x}^k - \frac{d\ln T}{dt}\dot{x}^l = 0$$

$$\frac{d^2x^l}{dt^2} + \Gamma^l_{kj}\frac{dx^j}{dt}\frac{dx^k}{dt} - \frac{1}{T}\frac{dT}{dt}\frac{dx^l}{dt} = 0$$

If we use the arc length as a variable ds, then $T = \frac{ds}{dt} = 1/\frac{dt}{ds}$

$$\frac{dT}{dt}/T = \frac{-1}{(dt/ds)^2}\frac{d^2t}{ds^2}\frac{ds}{dt}\cdot\frac{dt}{ds} = -\frac{d^2t}{ds^2}\Big/\left(\frac{dt}{ds}\right)^2$$

$$\Rightarrow \left(\frac{dt}{ds}\right)^2\frac{d^2x^l}{dt^2} + \Gamma^l_{kj}\frac{dx^j}{ds}\frac{dx^k}{ds} + \frac{d^2t}{ds^2}\frac{dx^l}{dt} = 0$$

$$\left(\frac{dt}{ds}\right)^2\frac{d^2x^l}{dt^2} = \frac{dt}{ds}\frac{d}{ds}\left[\frac{d}{dt}x^l\right] = \frac{dt}{ds}\left[\frac{d}{ds}\left(\frac{d}{ds}x^l\frac{ds}{dt}\right)\right] = \frac{d^2}{ds^2}x^l + \frac{dt}{ds}\frac{d}{ds}x^l\frac{d}{ds}\left(\frac{1}{dt/ds}\right)$$

$$= \frac{d^2}{ds^2}x^l - \frac{d}{ds}x^l\frac{dt}{ds}\frac{1}{\left(\frac{dt}{ds}\right)^2}\frac{d^2t}{ds^2} = \frac{d^2}{ds^2}x^l - \frac{dx^l}{dt}\frac{d^2t}{ds^2}$$

$$\Rightarrow \quad \boxed{\frac{d^2x^l}{ds^2} + \Gamma^l_{kj}\frac{d x^k}{ds}\frac{dx^j}{ds} = 0}$$

If on a curved surface without force tangent to the surface, the speed is a constant, hence $s = vt$. Hence, the trajectory $x^i(t)$ through the Lagrange equation, is the same of the geodesic equation $x^i(s)$.

✡ Gradient, divergence, and curl

In a curved manifold, we can use covariant derivatives to generalize the gradient, divergence, and curl. For a scalar function,

$$D_\mu \phi(x) = \partial_\mu \phi(x) \quad \text{— the gradient remains unchanged}$$

The covariant derivative of a contravariant vector

$$D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\sigma} A^\sigma \longrightarrow D_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\mu\sigma} A^\sigma$$

$$= \partial_\mu A^\mu + \frac{1}{\sqrt{g}} \partial_\sigma \sqrt{g}\, A^\sigma$$

$$\Rightarrow \boxed{D_\mu A^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, A^\mu \right)}$$

Gauss's theorem

$$\int d^dx \sqrt{g}\, D_\mu A^\mu = \int d^dx \sqrt{g} \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, A^\mu) = \int d^dx\, \partial_\mu (\sqrt{g}\, A^\mu)$$

$$= \oiint ds^\mu \sqrt{g}\, A^\mu = 0, \quad (\text{if } A^\mu \text{ vanishes on the boundary}).$$

Laplacian:

$$D^\mu \phi(x) = g^{\mu\nu} D_\nu \phi(x) = g^{\mu\nu} \partial_\nu \phi = \partial^\mu \phi(x)$$

$$\boxed{D_\mu D^\mu \phi = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g}\, \partial^\mu \phi(x) \right)}$$

The divergence of a tensor

$$D_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma}$$

$$= \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, T^{\mu\nu}) + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma}$$

if $T^{\mu\sigma}$ is antisymmetric, since $\Gamma^\nu_{\mu\sigma}$ is symmetric

$$\Rightarrow \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} = 0$$

$$\Rightarrow \boxed{D_\mu T^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, T^{\mu\nu})}$$

• Curl: $\boxed{\begin{aligned} D_\mu A_\nu - D_\nu A_\mu &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}}$ $= \partial_\mu A_\nu - \Gamma^\lambda_{\mu\nu} A_\lambda - (\partial_\nu A_\mu - \Gamma^\lambda_{\nu\mu} A_\lambda)$

• For anti-symmetric tensor $A_{\mu\nu}$, we have

$$\begin{aligned} D_\lambda A_{\mu\nu} + D_\mu A_{\nu\lambda} + D_\nu A_{\lambda\mu} &= \partial_\lambda A_{\mu\nu} - \Gamma^\delta_{\lambda\mu} A_{\delta\nu} - \Gamma^\delta_{\lambda\nu} A_{\mu\delta} \\ &+ \partial_\mu A_{\nu\lambda} - \Gamma^\delta_{\mu\nu} A_{\delta\lambda} - \Gamma^\delta_{\mu\lambda} A_{\nu\delta} \\ &+ \partial_\nu A_{\lambda\mu} - \Gamma^\delta_{\nu\lambda} A_{\delta\mu} - \Gamma^\delta_{\nu\mu} A_{\lambda\delta} \end{aligned}$$

notice $A_{\mu\nu} = -A_{\nu\mu}$

$$= \partial_\lambda A_{\mu\nu} + \partial_\mu A_{\nu\lambda} + \partial_\nu A_{\lambda\mu}$$

Hence, the Bianchi identity of EM

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$ remains unchanged

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0. \checkmark$$

(*) Some results on total derivatives

① For a scalar function, we arrive at a covariant vector

$A_\mu = D_\mu \phi = \partial_\mu \phi$, such that an anti-symmetric rank-2 tensor

$$\boxed{D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \partial_\nu \phi - \partial_\nu \partial_\mu \phi = 0.}$$

② If we start with a covariant vector A_μ, we first construct 2-nd rank anti-symmetric tensor $A_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$

Then, a rank-3 anti-symmetric tensor

$$\boxed{A_{[\mu\nu\lambda]} = D_\lambda A_{\mu\nu} + D_\mu A_{\nu\lambda} + D_\nu A_{\lambda\mu} = 0}$$

③ This can be generalized to high ranks.

- The rank-n fully anti-symmetric tensors in n-dimensions $T_{[\mu_1 \cdots \mu_n]}$. $\Rightarrow$ $\varepsilon^{\mu_1 \cdots \mu_n} T_{[\mu_1 \cdots \mu_n]} = \phi(x)$ is a scalar, hence

$$\boxed{T_{[\mu_1 \cdots \mu_n]} = \varepsilon_{\mu_1 \cdots \mu_n} \phi(x)}$$

(*) Closed and exact forms

Consider an fully anti-symmetric tensor $A_{\lambda_1 \cdots \lambda_{n-1}}$ define

$$D_{[\lambda_1} A_{\lambda_2 \cdots \lambda_n]} = D_{\lambda_1} A_{\lambda_2 \cdots \lambda_n} \pm D_{\lambda_2} A_{\lambda_3 \cdots \lambda_1} + D_{\lambda_3} A_{\lambda_4 \cdots \lambda_2} \pm D_{\lambda_4} A_{\lambda_5 \cdots}$$

"$\pm$" applies for the rotation of the n-object is even or odd.

① If $\boxed{D_{[\lambda} A_{\mu\nu \cdots]} = 0}$, then we say $A_{\mu\nu \cdots}$ is closed.
closed form.

② If an anti-symmetric tensor $A_{\mu\nu \cdots}$ is exact, if it can be written as

$$\boxed{A_{\mu\nu \cdots} = D_{[\mu} T_{\nu \cdots]} = \partial_{[\mu} T_{\nu \cdots]}.}$$ exact form.

where T is an lower-rank fully anti-symmetric tensor.

For example, $\nabla \times \vec{E} = 0 \implies \vec{E} = -\nabla \phi$

$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, $F_{\mu\nu}$ is exact

- If a tensor is exat, then it is also closed, i.e $D_{[\lambda} F_{\mu\nu]} = 0$.
- Every closed tensor is locally exact at least, which admits a local potential.

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$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} \Rightarrow D_\mu \tilde{F}^{\mu\nu} = D_\mu\left(\frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}\right)$$

$$D_\mu \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\lambda\rho} D_\mu F_{\lambda\rho} = \frac{1}{2}\cdot\frac{1}{3}\epsilon^{\mu\nu\lambda\rho}[D_\mu F_{\lambda\rho} + D_\lambda F_{\rho\mu} + D_\rho F_{\nu\lambda}] = 0$$

- Covariant differentiation along a curve

Consider a curve $x^\mu(\tau)$ in a space, and a vector ξ^μ defined along such a curve $\xi^\mu(x(\tau)) = \xi^\mu(\tau)$. $\Rightarrow$ $\xi^\mu(\tau + d\tau) = \xi^\mu(\tau) + d\tau \frac{d\xi^\mu(\tau)}{d\tau}$

If we parallely transport $\xi^\mu(x(\tau))$ to $x(\tau+d\tau)$.

$$\xi^{*,\mu}(\tau+d\tau) = \xi^\mu(\tau) - \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\xi^\lambda(\tau)d\tau$$

$$\Rightarrow \xi^\mu(\tau+d\tau) - \xi^{*,\mu}(\tau+d\tau) = d\tau\left[\frac{d\xi^\mu(\tau)}{d\tau} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\xi^\lambda(\tau)\right] = d\tau\frac{D\xi^\mu(\tau)}{D\tau}$$

Hence $\frac{D}{D\tau}\xi^\mu(\tau) = \left[\frac{\partial\xi^\mu(\tau)}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda}\xi^\lambda(\tau)\right]\frac{dx^\nu}{d\tau} = D_\nu\xi^\mu\frac{dx^\nu}{d\tau}$

Similarly $\frac{D}{D\tau}T^{\mu_1\cdots\mu_n} = \frac{dx^\nu}{d\tau}D_\nu T^{\mu_1\cdots\mu_n}$

$$\frac{Dg^{\mu\nu}}{D\tau} = \frac{dx^\lambda}{d\tau}D_\lambda g^{\mu\nu} = 0, \quad \frac{Dg_{\mu\nu}}{D\tau} = \frac{dx^\lambda}{d\tau}D_\lambda g_{\mu\nu} = 0.$$

and $\frac{D\xi_\mu(\tau)}{D\tau} = \frac{D}{D\tau}(g_{\mu\nu}\xi^\nu) = g_{\mu\nu}\frac{D\xi^\nu}{D\tau} = g_{\mu\nu}D_\lambda\xi^\nu\frac{dx^\lambda}{d\tau}$

$$= D_\lambda\xi_\mu\frac{dx^\lambda}{d\tau}.$$