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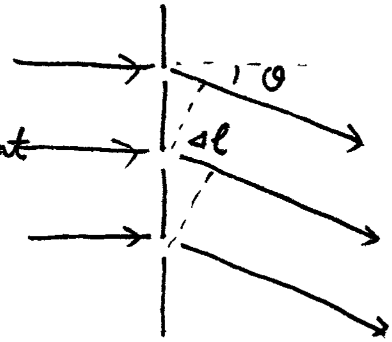
Lect 12 Diffraction (II)

* grating



a generalization of double slit experiment to multiple slit. The light distance difference between neighboring slits $\Delta l = d\sin\theta$

when $\Delta l = m\lambda$ or $\boxed{d\sin\theta_m = m\lambda}$ ← interference maximal.

The peak width of multiple-slit is much sharper than the case of double-slit. A deviation from θ_m, $\Delta\theta$ → many slits $(1 + e^{i\Delta\varphi} + e^{i2\Delta\varphi} + \cdots e^{i(N-1)\Delta\varphi})$ →

$$E_{tot} = E_{single\text{-}slit}\left[1 + e^{i\varphi} + \cdots e^{i(N-1)\varphi}\right]$$

where $\boxed{\varphi = \frac{2\pi}{\lambda} \cdot d\sin\theta}$

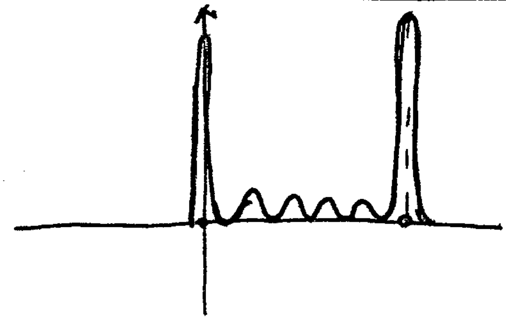
$$\Rightarrow E_{tot} = E_{single}\,\frac{1 - e^{iN\varphi}}{1 - e^{i\varphi}} = E_{single}\cdot e^{i\frac{N-1}{2}\varphi}\,\frac{\sin\frac{N}{2}\varphi}{\sin\frac{\varphi}{2}}$$

$$E_{single} = \frac{\sin\frac{\chi}{2}}{\frac{\chi}{2}} \quad \text{where} \quad \chi = \frac{2\pi}{\lambda} D\sin\theta$$

$$\Rightarrow \frac{|E_{tot}|^2}{|E_{tot}(\theta=0)|^2} = \underbrace{\left(\frac{\sin\frac{N}{2}\varphi}{\sin\frac{\varphi}{2}}\right)^2}_{\text{multiple-slit}} \underbrace{\left(\frac{\sin\frac{\chi}{2}}{\frac{\chi}{2}}\right)^2}_{\text{single slit factor}}$$

1. The maxima at $\varphi_m = 2m\pi$, where

$$\frac{\sin \frac{N}{2}\varphi_m}{\sin \frac{\varphi_m}{2}} = N$$

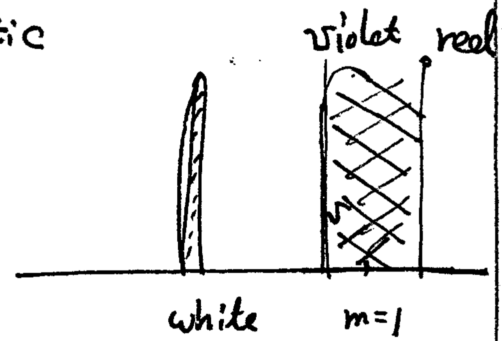


2. the width of the principle peaks $\varphi - \varphi_m = \pm \frac{2\pi}{N} \propto \frac{1}{N}$

☆ spectrum : if light is not monochromatic

the zeroth order peak is white

other high order peak develops dispersion



ex: determine the angular positions of the 1st, 2nd maxima for 400nm, 700nm, for grating 10,000 lines/cm.

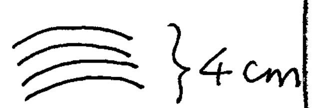
$d = 1\times10^{-2}/10^4 \text{ m} = 1\mu m$

$d \sin\theta_m = m\lambda \Rightarrow \sin\theta_{400} = \frac{m\lambda_{400}}{d} = \begin{cases} 0.4 & m=1 \Rightarrow \theta_{400} = 23.6^\circ \\ 0.8 & m=2 \quad \theta_{400} = 53.1^\circ \end{cases}$

$\sin\theta_{700} = \frac{m\lambda_{700}}{d} = \underset{0.7}{\overset{m=1}{}} \Rightarrow \theta_{400} = 44.4^\circ$

but second order doesnot exist

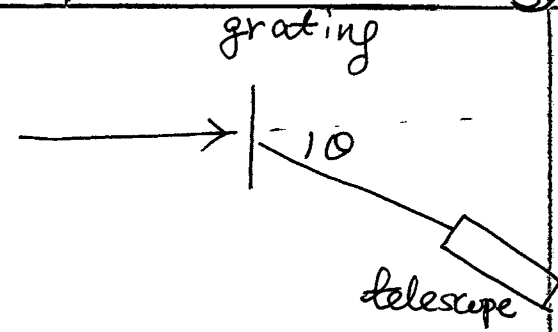
ex: CD. rainbow — reflection grating



$\underbrace{200-500}_{350}$ rev/min $\times$ 80 min $\approx$ 28,000 lines

$\Rightarrow d \approx \frac{4\text{ cm}}{28{,}000} \approx 1.4\,\mu m.$ if $\lambda = 550$nm $\boxed{d \sim 2\text{~}3\lambda}$.

* Spectrometer



$$\lambda = \frac{d}{m}\sin\theta$$

line spectrum : H.

grating 1×10^4 lines/cm — violet 24.2°
blue 25.7°
red 41.1°

$d = 1\ \mu m$ $\quad m=1 \Rightarrow \lambda = 1\mu m \sin 24.2° = 434\ nm$
$\sin 25.7 = 486\ nm$
$\sin 41.1 = 656\ nm$

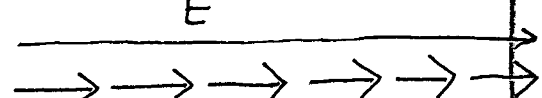
dark lines (absorpation) in sun's spectra.

* resolving power for a diffraction grating

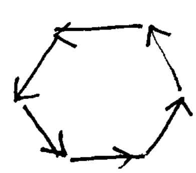
ex: 6-slits, N= 6

$$\frac{I(\theta)}{I(0)} = \left(\frac{\sin\frac{N}{2}\varphi}{\sin\frac{\varphi}{2}}\right)^2\left(\frac{\sin\chi}{\chi}\right)^2 \qquad \varphi = \frac{d\sin\theta}{\lambda}\cdot 2\pi \qquad \chi = \frac{D\sin\theta}{\lambda}2\pi$$

centeral maximum $\theta = 0,\ \delta = 0$



minimum $\theta = 60°$



E=0

- The first minimal occurs at $\frac{N}{2}\varphi = \pi$ or $\varphi = \frac{2\pi}{N}$

$$\frac{d \sin\theta \cdot 2\pi}{\lambda} = \frac{2\pi}{N} \Rightarrow d\,\theta \simeq \frac{\lambda}{N} \quad \text{or} \quad \theta_{min} = \frac{1}{N}\frac{\lambda}{d}$$

or the half-anglar width of the zeroth peak $\boxed{\Delta\theta_0 = \frac{\lambda}{Nd}}$

- the secondary peak

$$\left(\frac{\sin \frac{N\varphi}{2}}{\sin\frac{\varphi}{2}}\right)' = \frac{\cos\frac{N\varphi}{2} \cdot \frac{N}{2} \sin\frac{\varphi}{2} - \cos\frac{\varphi}{2} \cdot \frac{1}{2} \sin\frac{N\varphi}{2}}{\left(\sin\frac{\varphi}{2}\right)^2} = 0$$

$$\boxed{N \tan \varphi/2 = \tan N\varphi/2}$$

Since N is large $\Rightarrow \frac{N\varphi}{2} \approx \frac{3}{2}\pi$

$$\frac{I_{second}}{I_0} \approx \left(\frac{1}{N \sin\frac{3\pi}{2N}}\right)^2 \xrightarrow[N\to\infty]{\sim} \left(\frac{1}{\frac{3\pi}{2}}\right)^2 \sim \frac{1}{20}$$

- half width of m-th order principle peak

$$\Delta\varphi = \frac{2\pi}{N}, \quad \varphi = \frac{d\sin\theta}{\lambda} \cdot 2\pi$$

$$\Rightarrow \Delta\varphi = \frac{d\cos\theta_m}{\lambda} \cdot 2\pi\, \Delta\theta_m$$

or $\boxed{\Delta\theta_m = \frac{\lambda}{2\pi d \cos\theta_m} \cdot \frac{2\pi}{N} = \frac{\lambda}{N d\cos\theta_m}}$

resolving power

$$\Delta\theta_m = \frac{\lambda}{Nd\cos\theta_m}$$ half-width for m-th peak for λ

for another wavelength $\lambda + d\lambda$, according to $d\sin\theta_m = m\lambda$

$$\Rightarrow d\cos\theta_m \Delta\theta_m' = m\Delta\lambda$$

. The resolving limit is that the peak of $\lambda + d\lambda$, lies outside the half-width of λ, ie $\Delta\theta_m' = \frac{m\Delta\lambda}{d\cos\theta_m} > \Delta\theta_m = \frac{\lambda}{Nd\cos\theta_m}$

$$\Rightarrow R = \frac{\lambda}{\Delta\lambda} < Nm \quad \text{or} \quad (\Delta\lambda)_{min} = \frac{\lambda}{R}.$$

Sodium Yellow doublet $\lambda_1 = 589\,nm$ $\lambda_2 = 589.59\,nm$

grating 7500 line / cm

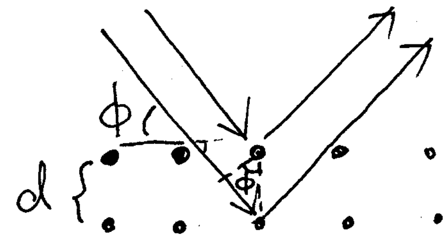
(1) $d\sin\theta = m\lambda \Rightarrow |m| < \frac{d}{\lambda} \approx \frac{\frac{1\,cm}{7500}}{589\,nm} \approx 2.26$

$\Rightarrow$ maximal value of m is 2.

(2) $R = \frac{\lambda}{\Delta\lambda} \approx \frac{589}{0.59} \approx 1000 = N m$

set $m=2 \Rightarrow N = 500$. Thus the minimal with of the grating $Nd = 500 \times \frac{1\,cm}{7500} \approx 0.067\,cm$

Crystal as grating — X-ray



$$\boxed{2d\sin\phi = m\lambda \qquad m = 1, 2, 3}$$

Bragg scattering

X-ray. Roentgen ~

10^{-2} nm ~ 10 nm
wavelength

many possible crystal planes

Ex $\lambda = 1\,\text{Å}$, NaCl, second order peak at $\phi = 21°$

$$\Rightarrow 2d\sin 21° = 2\lambda \Rightarrow d = \frac{1\,\text{Å}}{\sin 21°} = 2.8\,\text{Å}$$