Lect 8 — Hubbard model (Ogatta-Shiba)

① Hubbard model in 1D

$$H = -t \sum_{\langle ij \rangle, \sigma} \left(C_{i\sigma}^{+} C_{j\sigma} + h.c. \right) + U \sum_{i} n_{i\uparrow} n_{i\downarrow}$$

N is the # of particles with coordinates $x_1, \cdots x_N$

Among them, $\underbrace{x_1, \cdots x_M}_{M}$ sites with spin down; $x_{M+1}, \cdots x_N$ sites are with spin up.

The Bethe Ansatz wavefunction

$$f(x_1, \cdots x_N) = \sum_{Q} \theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N}) \left(\sum_{P} A(Q, P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}} \right)$$

where $P = (P_1, P_2, \cdots P_N)$, $Q = (Q_1, \cdots Q_N)$

momentum permutation — coordinate permutation

The values of $(k_1, \cdots k_N)$ need to be determined from the BA equation and the energy $E = -2t \sum_{j=1}^{N} \cos k_j$. The BA equation for the 1D Hubbard model is a straightforward generalization of Yang's solution:

k_j satisfies:

$$e^{ik_j L} = \prod_{\beta=1}^{M} \frac{t \sin k_j - \Lambda_\beta + iU/4}{t \sin k_j - \Lambda_\beta - iU/4} \qquad j = 1, \cdots N$$

where $\Lambda_1, \cdots \Lambda_M$ are a set of unequal #'s satisfying

$$-\prod_{j=1}^{N} \frac{t \sin k_j - \Lambda_\alpha + iU/4}{t \sin k_j - \Lambda_\alpha - iU/4} = \prod_{\beta=1}^{M} \frac{\Lambda_\beta - i\Lambda_\alpha + iU/2}{\Lambda_\beta - i\Lambda_\alpha - iU/2} \qquad \alpha = 1, 2, \cdots M.$$

Then Schrödinger Eq:

$$H|\psi\rangle = E|\psi\rangle \Rightarrow$$

$$-t\sum_i \left[f(x_1, \cdots x_i+1, \cdots x_N) + f(x_1, \cdots x_i-1, \cdots x_N)\right]$$

$$+ U\sum_{k<k'} \delta_{x_k, x_{k'}} f(x_1, x_2, \cdots x_N) = E f(x_1, x_2, \cdots x_N)) \quad (*)$$

where $\delta_{x_k, x_{k'}}$ means $\delta_{x_k, x_{k'}}$ and also $\delta_{\sigma_k \downarrow}\delta_{\sigma_{k'}\uparrow}$

(since the convention that particle indices $1, 2, \cdots M$ for spin $\downarrow$, and $M+1, \cdots N$ for spin $\uparrow$, the spin-index is not marked explicitly)

plug in: $f(x_1, \cdots x_N) = \sum_Q \left[\theta(x_{Q_1} < \cdots x_{Q_N}) \sum_P A(Q,P)\, e^{i\sum_{j=1}^{N} k_{P_j} x_{Q_j}}\right]$.

$\boxed{\text{example:}}$

domain Q: $0 < x_{Q_1} < \underline{x_{Q_2} < x_{Q_3}} < \cdots < x_{Q_N} < L$

Q': $0 < x_{Q_1} < \underline{x_{Q_3} < x_{Q_2}} < \cdots < x_{Q_N} < L$

Set $x_{Q_2} = x_{Q_3} = x$, f continuous $\Rightarrow$

$\boxed{f_Q \text{ is the WF in the domain of } Q}$

$$f_Q(\cdots \underset{x_{Q_2}}{\underset{\uparrow}{x}} \cdots \underset{x_{Q_3}}{\underset{\uparrow}{x}} \cdots) = f_{Q'}(\cdots \underset{x_{Q'_3}}{\underset{\uparrow}{x}}, \cdots \underset{x_{Q'_2}}{\underset{\uparrow}{x}} \cdots)$$

$$\Rightarrow A(Q,P)\, e^{ik_{P_2}x_{Q_2} + k_{P_3}x_{Q_3}} + A(Q,P')\, e^{ik_{P'_2}x_{Q_2} + ik_{P'_3}x_{Q_3}}$$

$$= A(Q',P)\, e^{ik_{P_2}x_{Q'_2} + ik_{P_3}x_{Q'_3}} + A(Q',P')\, e^{ik_{P'_2}x_{Q'_2} + ik_{P'_3}x_{Q'_3}}$$

where $P = (P_1, \underline{P_2, P_3} \cdots P_N)$ and $P' = (P_1\ \underline{P_3\ P_2} \cdots P_N)$

$\Rightarrow$ set $x_{Q_2} = x_{Q_3} = x_{Q'_2} = x_{Q'_3} = x$, $\Rightarrow A(Q,P) + A(Q,P') = A(Q',P) + A(Q',P')$

or $\boxed{A(Q,P') - A(Q',P) = A(Q',P') - A(Q,P)}$ (***)

when $x_{Q_2} = x_{Q_3} = x$, and others non-equal

$$-t\Big[\underbrace{f_{Q'}(\cdots x+1, \cdots x \cdots)}_{x_{Q_2} > x_{Q_3}} + \underbrace{f_{Q}(\cdots x-1, \cdots x \cdots)}_{x_{Q_2} < x_{Q_3}} + f_Q(\cdots x, \cdots x+1 \cdots)$$

x_{Q_2} position Q_3 position

$$+ f_{Q'}(\cdots x, \cdots x-1, \cdots)\Big] - t\sum_{i \neq Q_2, Q_3} \left[f_Q(\cdots x_i+1, \cdots) + f_Q(\cdots x_i - 1 \cdots)\right]$$

$$+ u f_Q(x_1, \cdots x \cdots x \cdots) = E f_Q(x_1, \cdots x \cdots x \cdots). \quad (*)$$

Q_2 position Q_3 position

Q, Q' are permutation, but Q_2, Q_3 are just # of indices.

The above Eq mainly fouses on domain Q; in this domain, the hopping of Q_2 and Q_3'th particle can lead to the domain change

If all the coordinates in Q, are non-equal to each other

$\rightarrow$ $-t\sum_i f_1(\cdots x_i+1, \cdots) + f_1(\cdots x_i - 1 \cdots) = E f(x_1 \cdots x_N)$

$\rightarrow$ extent to $x_{Q_2} = x_{Q_3} = x$, we have

$$-t\Big[f_Q(\cdots x+1 \cdots x \cdots) + f_Q(\cdots x \cdots x-1 \cdots) + f_Q(\cdots x \cdots x+1)$$

Q_2's position Q_3's postion

$$+ f_Q(\cdots x-1, \cdots x)\Big] - t\sum_{i \neq Q_2 Q_3} f_Q(\cdots x_i+1 \cdots) + f_Q(\cdots x_i - 1 \cdots)$$

$$= E f_Q(x_1 \cdots x \cdots x \cdots x_N) \quad (**)$$

Q_2's posit Q_3's position

Among them, in $f_1(\cdots x+1, \cdots x \cdots)$ and $f_1(\cdots x, \cdots x-1 \cdots)$

actually $x_{Q_3} < x_{Q_2}$ now, which should not be in Q, but we just extend the expressin in f_Q to $f_{Q'}$.

④

Take the difference between Eq (*) and (**)

in the sense of continuation

$$\boxed{\begin{aligned}&-t\Big[f_Q(\cdots \underset{\uparrow Q_2}{x+1}, \cdots \underset{\uparrow Q_3}{x} \cdots) + f_Q(\cdots \underset{\uparrow Q_2}{x}, \cdots \underset{\uparrow Q_3}{x-1}, \cdots) \\ &- f_{Q'}(\cdots \underset{\uparrow Q_3'}{x+1}, \cdots \underset{\uparrow Q_2'}{x} \cdots) - f_{Q'}(\cdots \underset{\uparrow Q_3'}{x} \cdots \underset{\uparrow Q_2'}{x-1} \cdots)\Big] - u f_Q(\cdots x \cdots x) = 0\end{aligned}}$$

plug in $f_Q = \sum_P A(Q,P)\, e^{i\sum_{j=1}^{N} x_{Q_j} k_{P_j}}$

$$\begin{aligned}\Rightarrow -t\Big[& A(Q,P)\, e^{ik_{P_2}(x+1) + ik_{P_3}x} + A(Q,P')\, e^{ik_{P_3}(x+1) + ik_{P_2}x} \\ &+ A(Q,P)\, e^{ik_{P_2}x + ik_{P_3}(x+1)} + A(Q,P')\, e^{ik_{P_3}x + ik_{P_2}(x-1)}\Big] \\ +t\Big[& A(Q',P)\, e^{ik_{P_2}x + ik_{P_3}(x+1)} + A(Q',P')\, e^{ik_{P_3}x + ik_{P_2}(x+1)} \\ &+ A(Q',P)\, e^{ik_{P_2}(x-1) + ik_{P_3}x} + A(Q',P')\, e^{ik_{P_3}(x-1) + ik_{P_2}x}\Big] \\ -u\Big[& A(Q,P)\, e^{ik_{P_2}x + ik_{P_3}x} + A(Q,P')\, e^{ik_{P_3}x + ik_{P_2}x}\Big] = 0\end{aligned}$$

$$\begin{aligned}\Rightarrow &- A(Q,P)\left(e^{ik_{P_2}} + e^{-ik_{P_3}}\right) - A(Q,P')\left(e^{ik_{P_3}} + e^{-ik_{P_2}}\right) \\ &+ A(Q',P)\left(e^{-ik_{P_2}} + e^{ik_{P_3}}\right) + A(Q',P')\left(e^{-ik_{P_3}} + e^{ik_{P_2}}\right) \\ &- \frac{u}{t}\left[A(Q,P) + A(Q,P')\right] = 0\end{aligned}$$

$$\boxed{\begin{aligned}\Rightarrow &\left[A(Q,P) - A(Q',P')\right]\left(e^{ik_{P_2}} + e^{-ik_{P_3}}\right) \\ &+ \left[A(Q,P') - A(Q',P)\right]\left(e^{ik_{P_3}} + e^{-ik_{P_2}}\right) + u/t\left(A(Q,P) + A(Q,P')\right) = 0\end{aligned}} \quad (****)$$

use Eqs (***) and (****), we can eliminate $A(Q',P)$

$$\left[A(Q,P) - A(Q',P')\right]\left(e^{ik_{P_2}} + e^{-ik_{P_3}}\right) + \left(A(Q',P') - A(Q,P)\right)\left(e^{ik_{P_3}} + e^{-ik_{P_2}}\right) + u/t\,\big(A(Q,P) + A(Q,P')\big)$$

$$A(Q,P)\left[e^{ik_{P_2}} - e^{-ik_{P_2}} + U/t\right] = -A(Q',P')\left(e^{ik_{P_3}} - e^{-ik_{P_3}} + e^{-ik_{P_2}} - e^{ik_{P_3}}\right.$$
$$\left. + e^{-ik_{P_3}} - e^{ik_{P_3}} \right) - U/t\, A(Q,P')$$

$$A(Q,P)\left[\sin k_{P_2} - \sin k_{P_3} - \frac{U}{2t} i\right] = A(Q',P')\left[\sin k_{P_2} - \sin k_{P_3}\right] + \frac{U}{2t} i\, A(Q,P')$$

$$A(Q,P) = \frac{(\sin k_{P_2} - \sin k_{P_3}) A(Q',P') + \frac{U}{2t} i\, A(Q,P')}{\sin k_{P_2} - \sin k_{P_3} - \frac{U}{2t} i}$$

$$= Y^{23}_{P_3 P_2}\, A(Q,P')$$

with $$Y^{23}_{P_3P_2} = \frac{(\sin k_{P_2} - \sin k_{P_3})\, P_{Q_2Q_3} + \frac{U}{2t} i}{(\sin k_{P_2} - \sin k_{P_3} - \frac{U}{2t} i)}$$

More generally,

$$\boxed{\begin{gathered} A(Q,P) = Y^{i,i+1}_{n,m}\, A(Q,P') \\ \text{where } Y^{i,i+1}_{n,m} = \frac{P_{i,i+1} - X_{nm}}{1 + X_{nm}} \quad \text{where } X_{nm} = \frac{iU/2t}{\sin k_n - \sin k_m} \end{gathered}}$$

(*) $P_{i,i+1}$ is the exchange acting on $Q = (Q_1, \cdots Q_i, Q_{i+1}, \cdots Q_N)$

it acts on the ith and i+1 th positions rather than the indices of $i, i+1$

$$P_{i,i+1}(Q_1 \cdots Q_N) = Q' = (Q_1, \cdots Q_{i+1}, Q_i, \cdots Q_N)$$

(*) The P' on the RHS is a permutation for momentum

$$P' = (P_1' = P_1, \cdots \underline{P_i' = n,\ P_{i+1}' = m}, \cdots P_N' = P_N)$$

and the LHS $P = (P_1, P_2 \cdots \underline{P_i = m,\ P_{i+1} = n}, \cdots P_N)$

① If we have the information for $A(Q, P' = I)$ for all the Q's.

we know $A(Q, P)$ for all Q, if P is a nearest neighbor exchange.

$P = (1\,2; i{+}1\,i; \cdots n)$

$$A(Q, P) = \frac{1}{1 + X_{i,i+1}} A(Q', P' = I) - \frac{X_{i,i+1}}{1 + X_{i,i+1}} A(Q, P')$$

and $Q' = (Q_1, \cdots Q_{i+1}, Q_i, \cdots)$.

If we know $A(Q, P')$ with $P' = (\cdots \underset{i\text{th}}{n}\ \underset{i{+}1\text{th}}{m} \cdots)$, for all Q's.

then for $P = (\cdots m\, n \cdots)$, we have

$$\boxed{A(Q, P) = \frac{1}{1 + X_{nm}} A(Q', P') - \frac{X_{nm}}{1 + X_{nm}} A(Q, P')}$$

where $Q' = (Q_1, \cdots Q_{i+1}, Q_i \cdots)$. and $X_{nm} = \dfrac{i\,u/2t}{\sin k_n - \sin k_m}$.

Since All the permutation P can be arrived by starting with $P' = I$ by exchanging nearest neighbors, we have all the information $A(Q, P)$.

Or conversely, we can convert all $A(Q, P)$ back to $A(Q, I)$.

② The above relation can also be written as

$$\boxed{A(Q, P) = \frac{1 - X_{nm} P_{i,i+1}}{1 + X_{nm}} A(Q', P')}$$

Denote

$$X_{nm}^{i,i+1} = \frac{1 - X_{nm} P_{i\,i+1}}{1 + X_{nm}}$$

Apply it

$$\frac{1 - x_{1j} P_{1j}}{1 + x_{1j}} \frac{1 - x_{2j} P_{2j}}{1 + x_{2j}} \cdots \frac{1 - x_{j-1,j} P_{j-1,j}}{1 + x_{j-1,j}} A(Q_1 \cdots Q_N, 1\,2 \cdots j{-}1\, j \cdots N)$$

$$= A(Q_j\, Q_1 \cdots Q_{j-1}, Q_{j+1}, \cdots Q_N;\ j\,1 \cdots j{-}1\, j{+}1 \cdots N)$$

$$\frac{1-x_{j,N}P_{j,N}}{1+x_{j,N}}\cdots\frac{1-x_{j,j+2}P_{j,j+2}}{1+x_{j,j+2}}\frac{1-x_{j,j+1}P_{j,j+1}}{1+x_{j,j+1}}A(Q_1\cdots Q_N,\ 1\,2\cdots j,j+1\cdots N)$$

$$= A(Q_1\cdots Q_{j-1}Q_{j+1}\cdots Q_N Q_j,\ 1\cdots j-1,j+1\cdots N j)$$

The periodical boundry condition $\longrightarrow$

$$A(Q_1\cdots Q_N;\ P_1\cdots P_N) = A(Q_2Q_3\cdots Q_N Q_1,\ P_2\cdots P_N P_1)\,e^{ik_{P_1}L}$$ $\leftarrow$ see Lect 5

$$\rightarrow A(Q_j Q_1\cdots Q_{j-1}Q_{j+1}\cdots Q_N,\ j\,1\cdots j-1\,j+1\cdots N)$$

$$= A(Q_1\cdots Q_{j-1}Q_{j+1}\cdots Q_N Q_j,\ 1\cdots j-1\,j+1\cdots N j)\,e^{ik_jL}$$

$\Rightarrow$ $\boxed{\text{define } X_{ij} = \frac{1-x_{ij}P_{ij}}{1+x_{ij}}}$ $\Rightarrow X_{ij}^{-1} = X_{ji}$, (note that $x_{ij} = -x_{ji}$)

$$X_{1j}X_{2j}\cdots X_{j-1j}\,A(Q_1\cdots Q_N,\ I) = e^{ik_jL}X_{jN}\cdots X_{j,j+1}\,A(Q_1\cdots Q_N,\ I)$$

$$\Rightarrow \boxed{X_{j+1,j}\cdots X_{N-1j}X_{Nj}X_{1j}X_{2j}\cdots X_{j-1,j}\,A(Q,I) = e^{ik_jL}A(Q,I)}$$

We view $A(Q,I)$ as a colowmnar vector of Q, and the operator P_{ij} as an exchange of Q_i and Q_j, i.e. the "i" and "j" th positions of Q.

Define $A(Q,I) = (-)^Q\chi(Q)$, then since $P_{ij}\,A(Q_1\cdots Q_N,I) = A(\cdots Q_j\cdots Q_i\cdots)$

$$P_{ij}\,\chi(Q) = (-)^Q P_{ij}\,A(Q_1\cdots Q_N,I) = (-)^Q A[Q_1\cdots Q_j\cdots Q_N,I] = (-)^Q(-)^{Q'}\chi(Q')$$

$$= -\chi(Q') \Rightarrow$$

$$\boxed{X'_{j+1,j}X'_{j+2,j}\cdots X'_{N,j}X'_{1,j}\cdots X'_{j-1,j}\,\chi(Q) = e^{ik_jL}\chi(Q)}$$

(*) How many independent components in $\chi(Q)$?

It looks that there are $N!$ configurations. The permutations among the alike spins, will not generate independent configurations, i.e. these amplitude can be achieved by using Fermi statistics

$$f(x_{T1}, x_{T2} \cdots x_{TM}) = (-)^T f(x_1, \cdots x_M; x_{M+1}, \cdots x_N)$$

where T is a permutation that only takes place among spins alike.

Then
$$f(x_{T1}, \cdots x_{TM}) = \sum \Big(\theta(x_{T[Q1]} < x_{T[Q2]} < \cdots x_{T[QN]})$$
$$\cdot \sum_P A(Q, P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{T[Q_j]}} \Big) \qquad \text{set } TQ = Q' \Rightarrow Q = T^{-1}Q'$$
$$= \sum_{Q'} \Big(\theta(x_{Q'_1} < \cdots x_{Q'_N}) \sum_P A(T^{-1}Q', P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q'_j}} \Big)$$
$$= \sum_Q \Big(\theta(x_{Q_1} < \cdots x_{Q_N}) \sum_P A(T^{-1}Q, P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}} \Big)$$

$$\Rightarrow \boxed{A(T^{-1}Q, P) = (-)^T A(Q, P)} \leftarrow f(x_1 \cdots x_N) = \sum_Q \theta(x_{Q_1} < \cdots < x_{Q_N}) \Big(\sum_P A(Q, P)\, e^{i \sum_{j=1}^{N} k_{P_j} x_{Q_j}} \Big).$$

hence $\chi(Q) = (-)^Q A(Q, 1)$ only has $\frac{N!}{M!(N-M)!}$ independent components! Permutations among the same spin, gives the same $\chi(Q)$. Hence

$\chi(Q)$ is characterized by the location of spin down particles $(y_1 < y_2 < \cdots y_M)$ in the permutation of $(Q_1, Q_2 \cdots Q_N)$.

Please note that $y_1 \cdots y_M$ are NOT the site coordinates of spin $\downarrow$ particles.

Say for $Q = (1\ 2, \cdots N) \Rightarrow y_1 = 1,\ y_2 = 2, \cdots y_M = M$

and $Q = (2\ 1, \cdots N)$ (remember $x_1, \cdots x_M$ are spin $\downarrow$)

the "M+1" position

but for $Q = (1\ 2, \cdots M-1, M+1, M, M+2, \cdots N) \longrightarrow y_1 = 1, \cdots y_{M-1} = M-1$

$\downarrow\ \downarrow\ \ \downarrow\ \uparrow\ \downarrow\ \uparrow \cdots \uparrow$ $\quad y_M = M+1$

C.N. Yang reduces the eigenvalue problem of $\chi(Q)$ to a Heisenberg chain problem $\Rightarrow$

$$\chi = \phi(y_1, \cdots y_M) = \sum_P A_P F(\Lambda_{P_1}, y_1) F(\Lambda_{P_2}, y_2) \cdots F(\Lambda_{P_M}, y_M)$$

where P is a permutation for spin $\downarrow$ particles among $(1\ 2 \cdots M)$

where
$$F(\Lambda, y) = \prod_{j=1}^{y-1} \frac{t \sin k_j - \Lambda + iU/4}{t \sin k_{j+1} - \Lambda - iU/4}$$

and
$$A_P = (-)^P \prod_{i<j} (\Lambda_{P_i} - \Lambda_{P_j} - iU/2).$$

So far, I have presented the BA wavefunction, without proof.

Next, we prove the facterization of the BA wavefunction

① At the limit $U/t \to \infty$, $Y^{i,i+1}_{n,m} = -1$

$$A(Q,P) = (-)A(Q,P') = (-)^P A(Q,I) = (-)^{P+Q}\chi(Q)$$

$$f(x_1 \cdots x_N) = \sum_Q \theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N}) \sum_P A(Q,P)\, e^{i\sum_{j=1}^{N} k_{P_j} x_{Q_j}}$$

$$= \sum_Q \left(\theta(x_{Q_1} < \cdots < x_{Q_N}) (-)^Q \chi(Q) \sum_P (-)^P e^{i\sum_{j=1}^{N} k_{P_j} x_{Q_j}} \right)$$

$$\chi(Q) = \phi(y_1, y_2, \cdots y_M)$$

$$\sum_P (-)^P e^{i\sum_{j=1}^{N} k_{P_j} x_{Q_j}} = \begin{bmatrix} e^{ik_1 x_{Q_1}} & e^{ik_1 x_{Q_2}} & \cdots & e^{ik_1 x_{Q_N}} \\ e^{ik_2 x_{Q_1}} & \cdots & & e^{ik_2 x_{Q_N}} \\ \vdots & & & \vdots \\ e^{ik_N x_{Q_1}} & \cdots & & e^{ik_N x_{Q_N}} \end{bmatrix} = \det M[Q]$$

$$= (-)^Q \begin{bmatrix} e^{ik_1 x_1} & e^{ik_1 x_2} & \cdots & e^{ik_1 x_N} \\ e^{ik_2 x_1} & e^{ik_2 x_2} & \cdots & e^{ik_2 x_N} \\ \vdots & & & \\ e^{ik_N x_1} & \cdots & & e^{ik_N x_N} \end{bmatrix}$$

$$f(x_1 \cdots x_N) = \sum_Q \left(\theta(x_{Q_1} < \cdots x_{Q_N}) (-)^Q \underbrace{\phi(y_1 \cdots y_M)} \underbrace{\det[e^{ik_i x_{Q_j}}]} \right)$$

only depends on the locations of spin down in the sequence of $(Q_1 \cdots Q_N)$.

spin less fermion WF

(*) Spin chain wavefunction

Recall the BA equation

$$-\prod_{j=1}^{N} \frac{\sin k_j - \Lambda_\alpha + iU/4}{\sin k_j - \Lambda_\alpha - iU/4} = \prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha + iU/2}{\Lambda_\beta - \Lambda_\alpha - iU/2} \qquad \alpha = 1, \cdots M$$

$$U \to \infty, \Rightarrow \quad \left(\frac{\Lambda_\alpha - iU/4}{\Lambda_\alpha + iU/4}\right)^N = -\prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha + iU/2}{\Lambda_\beta - \Lambda_\alpha - iU/2}$$

if we define $\lambda_\alpha = -\Lambda_\alpha/(U/2) \Rightarrow \left(\frac{\lambda_\alpha + i/2}{\lambda_\alpha - i/2}\right)^N = -\prod_{\beta=1}^{M}\left(\frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i}\right)$

This is precisely the BA equation for the isotropic Heisenberg model by identifying $\boxed{\lambda_\alpha = \frac{1}{2}\cot q_\alpha/2}$ — convention $0 \le q_\alpha < 2\pi$

$$\Rightarrow \boxed{e^{iq_\alpha} = \frac{\lambda_\alpha + i/2}{\lambda_\alpha - i/2} = \frac{-\Lambda_\alpha + iU/4}{-\Lambda_\alpha - iU/4}}$$

Then define $e^{i\phi_{\alpha\beta}} = -\frac{\lambda_\alpha - \lambda_\beta + i}{\lambda_\alpha - \lambda_\beta - i} \Rightarrow \cot\frac{\phi_{\alpha\beta}}{2} = \lambda_\alpha - \lambda_\beta$

$$= \frac{1}{2}\left(\cot\frac{q_\alpha}{2} - \cot\frac{q_\beta}{2}\right)$$

$$e^{i\phi_{\alpha\beta}} = -\frac{\Lambda_\beta - \Lambda_\alpha + \frac{U}{2}i}{\Lambda_\beta - \Lambda_\alpha - \frac{U}{2}i}$$ ← scattering phase shift

$$e^{i\phi_{\alpha\beta}} = -\frac{\Lambda_\alpha - \Lambda_\beta - U/2\, i}{\Lambda_\alpha - \Lambda_\beta + U/2\, i}$$

Now check wavefunctions

$$F(\Lambda_\alpha, y) \xrightarrow{U \to \infty} \prod_{j=1}^{y-1} \frac{-\Lambda_\alpha + iU/4}{-\Lambda_\alpha - iU/4} = e^{i(y-1)q_\alpha}$$

and $A_P = (-)^P \prod_{i<j} (\Lambda_{P_i} - \Lambda_{P_j} - iU/2)$ $\quad P = (P_1, P_2 \cdots P_M)$ permutation for M-downspin

$|A_P|^2 = \prod_{i<j} \{|\Lambda_i - \Lambda_j|^2 + |U/2|^2\}$ ← symmetric under permutations.

$$A_P = (-)^P \, e^{\frac{i}{2} \sum_{i<j} (\phi_{P_i P_j} - \pi)} \cdot \text{const}$$

amplitude

Then

$$\phi(y_1 \cdots y_M) = \sum_P (-)^P e^{i \sum_{i=1}^{M} q_{P_i}(y_i - 1) + \frac{i}{2} \sum_{i<j} \phi_{P_i P_j}}$$

$$= \boxed{\text{const}\, \underbrace{e^{i \sum_{i=1}^{M} q_i}}_{\text{constant}}} \cdot \underline{\sum_P \widetilde{A}_P \, e^{i \sum_{i=1}^{M} q_{P_i} y_i}}$$

consider two permutations $P: (\cdots P_\ell \; P_{\ell+1} \cdots)$, $P': (\cdots P_{\ell+1} \; P_\ell \cdots)$ } only differ a nearest neighbour ex

$$\Rightarrow \boxed{\underline{\frac{\widetilde{A}_{P'}}{\widetilde{A}_P}} = - e^{i \phi_{P_{\ell+1}, P_\ell}} = \frac{\lambda_{P_{\ell+1}} - \lambda_{P_\ell} + i}{\lambda_{P_{\ell+1}} - \lambda_{P_\ell} - i}}$$

from the change of even/odd-ness of P and P'

This is precisely the amplitudes of the BA wavefunction

for spin-1/2 Heisenberg chain!