§1. Conformal invariance :

Consider a coordinate system with the metric $ds^2 = g_{ab}(\xi) d\xi^a d\xi^b$,

and then we define a coordinate transformation $\eta^a(\xi)$, then

$$ds^2 = g_{ab}(\xi) \frac{\partial \xi^a}{\partial \eta^{a'}} \frac{\partial \xi^b}{\partial \eta^{b'}} d\eta^{a'} d\eta^{b'}$$

$$= g'_{a'b'}(\eta) \, d\eta^{a'} d\eta^{b'}$$

$$\boxed{\begin{gathered}\sqrt{g(\xi)} = \left| \frac{\partial \eta}{\partial \xi} \right| \sqrt{g(\eta)} \\ \int d\xi \sqrt{g(\xi)} = \int d\eta \frac{\partial \xi}{\partial \eta} \sqrt{g(\xi)} = \int d\eta \sqrt{g(\eta)}\end{gathered}}$$

$$\Rightarrow \boxed{g_{ab}(\xi) \frac{\partial \xi^a}{\partial \eta^{a'}} \cdot \frac{\partial \xi^b}{\partial \eta^{b'}} = g'_{a'b'}(\eta) \quad \text{with the relation } \eta^a = \eta^a(\xi)}$$

~~more~~ conveniently, change dummy variable $ab \to a'b'$ $\Rightarrow$

$$\boxed{g'_{ab}(\eta) = g_{a'b'}(\xi) \frac{\partial \xi^{a'}}{\partial \eta^a} \frac{\partial \xi^{b'}}{\partial \eta^b} \quad \text{with } \eta^a = \eta^a(\xi)}$$

"Conformal" means that $g'_{ab}(\eta) = \rho(\xi)\, g_{ab}(\xi)$, where $\rho(\xi)$ is a scalar function.

For a special case of $g_{a'b} = \delta_{ab}$ $\Rightarrow$

$$\frac{\partial \xi^\mu}{\partial \eta^a} \cdot \frac{\partial \xi^\mu}{\partial \eta^b} = \rho(\xi)\, \delta_{ab} \quad \rightarrow \text{take trace over } ab$$

$$\Rightarrow \rho(\xi) = \frac{1}{d} \frac{\partial \xi^\mu}{\partial \eta^a} \cdot \frac{\partial \xi^\mu}{\partial \eta^a}$$

$$\Rightarrow \boxed{\frac{\partial \xi^\mu}{\partial \eta^a} \cdot \frac{\partial \xi^\mu}{\partial \eta^b} = \frac{1}{d} \left[\frac{\partial \xi^\lambda}{\partial \eta^c} \frac{\partial \xi^\lambda}{\partial \eta^c} \right] \delta_{ab}}$$

This equation $\Rightarrow$ set $a=b=1$, and $a=1$, $b=2$,

$$\begin{cases} \dfrac{\partial \xi^1}{\partial \eta^1}\dfrac{\partial \xi^1}{\partial \eta^1} - \dfrac{\partial \xi^1}{\partial \eta^2}\dfrac{\partial \xi^1}{\partial \eta^2} = -\left(\dfrac{\partial \xi^2}{\partial \eta^1}\dfrac{\partial \xi^2}{\partial \eta^1} - \dfrac{\partial \xi^2}{\partial \eta^2}\dfrac{\partial \xi^2}{\partial \eta^2}\right) & (1) \\ \dfrac{\partial \xi^1}{\partial \eta^1}\dfrac{\partial \xi^1}{\partial \eta^2} = -\dfrac{\partial \xi^2}{\partial \eta^1}\dfrac{\partial \xi^2}{\partial \eta^2} & (2) \end{cases}$$

$$(1) + (2) i \times 2 \Rightarrow \left(\frac{\partial \xi^1}{\partial \eta^1} + i\frac{\partial \xi^1}{\partial \eta^2}\right)^2 = -\left(\frac{\partial \xi^2}{\partial \eta^1} + i\frac{\partial \xi^2}{\partial \eta^2}\right)^2$$

$$\Rightarrow \frac{\partial \xi^1}{\partial \eta^1} + i\frac{\partial \xi^1}{\partial \eta^2} = \pm i\left[\frac{\partial \xi^2}{\partial \eta^1} + i\frac{\partial \xi^2}{\partial \eta^2}\right]$$

$$\Rightarrow \boxed{\frac{\partial(\xi^1 \pm i\xi^2)}{\partial \eta^1} + i\frac{\partial(\xi^1 \pm i\xi^2)}{\partial \eta^2} = 0}$$

this is just Cauchy-Riemann condition

if we consider a group of continous transformation, which contains the identity $\begin{cases} \xi^1 = \eta^1 \\ \xi^2 = \eta^2 \end{cases}$, then we only have

$$\boxed{\frac{\partial(\xi^1 + i\xi^2)}{\partial \eta^1} + i\frac{\partial(\xi^1 + i\xi^2)}{\partial \eta^2} = 0}$$

On then other hand, $\dfrac{\partial \xi^\mu}{\partial \eta^a} \cdot \dfrac{\partial \xi^\mu}{\partial \eta^b} = \rho(\xi)\,\delta_{ab} \Rightarrow$

$$\frac{\partial \eta^a}{\partial \xi^\nu} \cdot \frac{\partial \eta^b}{\partial \xi^{\nu'}} \cdot \left[\frac{\partial \xi^\mu}{\partial \eta^a} \cdot \frac{\partial \xi^\mu}{\partial \eta^b}\right] = \rho(\xi)\,\delta_{ab}\frac{\partial \eta^a}{\partial \xi^\nu}\frac{\partial \eta^b}{\partial \xi^{\nu'}}$$

$$\delta_{\mu\nu}\delta_{\mu\nu'} = \delta_{\nu\nu'} = \rho(\xi)\frac{\partial \eta^a}{\partial \xi^\nu}\frac{\partial \eta^a}{\partial \xi^{\nu'}}$$

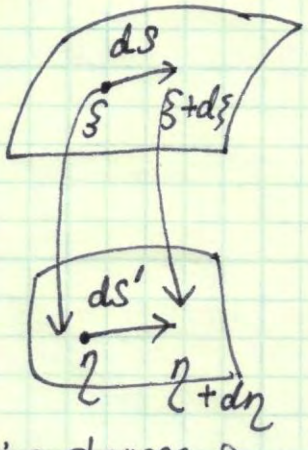
$$\Rightarrow \boxed{\frac{\partial \eta^a}{\partial \xi^\nu}\frac{\partial \eta^a}{\partial \xi^{\nu'}} = \rho'(\eta)\,\delta_{\nu\nu'}}$$

we also have

$$\boxed{\frac{\partial(\eta^1 + i\eta^2)}{\partial \xi^1} + i\frac{\partial(\eta^1 + i\eta^2)}{\partial \xi^2} = 0}$$

In the above, we have used the "passive" picture, i.e. we fix the length and uses two different coordinate systems. We can also use the "initiative" viewpoint. Consider a coordinate transformation

$$\xi^a \to \eta^a(\xi)$$



$$ds^2 = g_{ab}(\xi)\, d\xi^a d\xi^b$$

$$ds'^2 = g_{ab}(\eta)\, d\eta^a d\eta^b$$

$$= g_{ab}(\eta) \frac{\partial \eta^a}{\partial \xi^{a'}} \frac{\partial \eta^b}{\partial \xi^{b'}} d\xi^{a'} d\xi^{b'}$$

the same coordinate system, thus g_{ab} is the same, but the location changes from $\xi \to \eta$ and $d\xi^a \to d\eta^a$

$$= \bar{g}_{a'b'}(\eta) \frac{\partial \eta^{a'}}{\partial \xi^a} \frac{\partial \eta^{b'}}{\partial \xi^b} d\xi^a d\xi^b$$

It means that using $d\xi^a$, the metric for ds^2 changes to

$$\boxed{g_{a'b'}(\eta) \frac{\partial \eta^{a'}}{\partial \xi^a} \frac{\partial \eta^{b'}}{\partial \xi^b} = g'_{ab}(\xi)}$$

note the similarity and difference from the "passive" viewpoint!

Next, we will use the "initiative" viewpoint, and study infinitsmall transf

$$\eta^a(\xi) = \xi^a + \varepsilon^a(\xi)$$

$$\Rightarrow \quad g'_{ab}(\xi) = \left[g_{a'b'}(\xi) + \frac{\partial g_{a'b'}(\xi)}{\partial \xi^\lambda} \cdot \varepsilon^\lambda \right] \left[\delta_{a'a} + \frac{\partial \varepsilon^{a'}}{\partial \xi^a} \right] \left[\delta_{b'b} + \frac{\partial \varepsilon^{b'}}{\partial \xi^b} \right]$$

$$= g_{ab}(\xi) + \left[g_{a'b'} \frac{\partial \varepsilon^{a'}}{\partial \xi^a} \delta_{b'b} + g_{a'b'} \delta_{a'a} \frac{\partial \varepsilon^{b'}}{\partial \xi^b} + \frac{\partial g_{ab}(\xi)}{\partial \xi^\lambda} \varepsilon^\lambda \right]$$

$$d\, g_{ab}(\xi) = g_{a'b} \frac{\partial \varepsilon^{a'}}{\partial \xi^a} + g_{ab'} \frac{\partial \varepsilon^{b'}}{\partial \xi^b} + \frac{\partial g_{ab}(\xi)}{\partial \xi^\lambda} \varepsilon^\lambda = \alpha\, g_{ab}(\xi)$$

plug in $g_{ab} = \delta_{ab} \Rightarrow$

$$\frac{\partial \varepsilon^b}{\partial \xi^a} + \frac{\partial \varepsilon^a}{\partial \xi^b} = \rho(\xi)\, \delta_{ab} ,$$ ← take trace over a.b

$$\Rightarrow \boxed{\frac{\partial \varepsilon^b}{\partial \xi^a} + \frac{\partial \varepsilon^a}{\partial \xi^b} = \frac{2}{d} \left[\frac{\partial \varepsilon^\lambda}{\partial \xi^\lambda} \right] \delta_{ab}}$$

If $d = 2 \Rightarrow \begin{cases} \dfrac{\partial \varepsilon^1}{\partial \xi^1} = \dfrac{\partial \varepsilon^2}{\partial \xi^2} \\ \dfrac{\partial \varepsilon^1}{\partial \xi^2} = -\dfrac{\partial \varepsilon^2}{\partial \xi^1} \end{cases}$ set $\varepsilon = \varepsilon^1 + i \varepsilon^2$

$\bar{\varepsilon} = \varepsilon^1 - i \varepsilon^2$

$$\Rightarrow \begin{cases} \dfrac{\partial \varepsilon}{\partial \xi^1} + i \dfrac{\partial \varepsilon}{\partial \xi^2} = 0 \\ \dfrac{\partial \bar{\varepsilon}}{\partial \xi^1} - i \dfrac{\partial \bar{\varepsilon}}{\partial \xi^2} = 0 \end{cases} \Rightarrow$$

$\varepsilon(z)$ is a holomorphic func
$\bar{\varepsilon}(\bar{z})$ is a anti-holomorphic func

where $z = \xi_1 + i \xi_2$,
$\bar{z} = \xi_1 - i \xi_2$.

* In terms of z, and $\bar{z}$,

$$ds^2 = \frac{1}{2}\left[dz\, d\bar{z} + d\bar{z}\, dz\right] \Rightarrow \begin{cases} \eta_{zz} = \eta_{\bar{z}\bar{z}} = 0 \\ \eta_{z\bar{z}} = \eta_{\bar{z}z} = 1/2 \end{cases}$$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

$$\partial_x = \frac{\partial z}{\partial x} \partial_z + \frac{\partial \bar{z}}{\partial x} \partial_{\bar{z}} = \partial_z + \partial_{\bar{z}} \qquad \Rightarrow \partial_z = \frac{1}{2}[\partial_x - i \partial_y]$$

$$\partial_y = \frac{\partial z}{\partial y} \partial_z + \frac{\partial \bar{z}}{\partial y} \partial_{\bar{z}} = i(\partial_z - \partial_{\bar{z}}) \qquad \partial_{\bar{z}} = \frac{1}{2}[\partial_x + i \partial_y]$$

*) The holomorphic transformation can be generated by

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$$

Classical conformal algebra: $z \to z - \epsilon z^{n+1}$

$$\delta f(z) = f(z - \epsilon z^{n+1}) - f(z) = -\epsilon z^{n+1} \partial_z f = \epsilon \hat{l}_n f$$

the generator $\hat{l}_n = - z^{n+1} \partial_z$

Similarly $\bar{z} \to \bar{z} - \bar{\epsilon} \bar{z}^{n+1} \Rightarrow \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$

check commutation relation

$$[l_m, l_n] = z^{m+1} \partial_z (z^{n+1} \partial_z) - z^{n+1} \partial_z (z^{m+1} \partial_z)$$
$$= (n+1) z^{m+n+1} \partial_z - (m+1) z^{m+n+1} \partial_z = -(m-n) z^{m+n+1} \partial_z$$

$$\Rightarrow \boxed{\begin{array}{l} [l_m, l_n] = (m-n) l_{m+n}, \quad [\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n} \\ [l_m, \bar{l}_n] = 0 \end{array}}$$

$l_n = -z^{n+1} \partial_z$ is regular around $z=0$, only at $n \geq -1$

at $z \to \infty$, define $\omega = 1/z \Rightarrow l_n = -\omega^{-(n+1)} \partial_\omega \frac{\partial z}{\partial \omega}$

$$= \omega^{1-n} \partial_\omega .$$

l_n is regular at $z \to \infty$, only at $1-n \geq 0 \Rightarrow n \leq 1$

$\Rightarrow$ globally only $n = -1, 0, 1$ are well-defined!

$l_0 = -z\partial_z = -\frac{1}{2}(x+iy)(\partial_x - i\partial_y)$, $\bar{l}_0 = -\bar{z}\partial_{\bar{z}} = -\frac{1}{2}(x-iy)(\partial_x + i\partial_y)$

$$\Rightarrow \boxed{\begin{aligned} l_0 + \bar{l}_0 &= -(x\partial_x + y\partial_y) \quad \text{— scale transf} \\ l_0 - \bar{l}_0 &= +i(x\partial_y - y\partial_x) \quad \text{— rotation} \end{aligned}}$$

$n=-1$ $\quad l_{-1} = -\partial_z = -\frac{1}{2}(\partial_x - i\partial_y)$, $\bar{l}_{-1} = -\partial_{\bar{z}} = -\frac{1}{2}(\partial_x + i\partial_y)$

$$\boxed{l_{-1} + \bar{l}_{-1} = -\partial_x, \quad l_{-1} - \bar{l}_{-1} = i\partial_y} \text{ — translation}$$

$n=1$, $\quad l_1 = \partial_\omega \Rightarrow$ translation of ω (inverse)

$$\frac{1}{z} \to \frac{1}{z} + c \Rightarrow z = \frac{z}{cz+1}$$

using complex variable: the finite transformations can be written as:

$z \to \lambda z$, λ can be a complex number

$\to z + b$,

$\to \dfrac{z}{cz+1}$,

all these three transfs can be summarized as

$$z \to \frac{az+b}{cz+d} \quad \text{with } ad - bc = 1$$

check ① $c=0, d=1, a=1$, translation

② $b=0, c=0, d=1/a$, scale/rot

③ $a=1, b=0, d=1$, special conformal tran

if $a\,b\,c\,d$ flip the sign

$\dfrac{-az-b}{-cz-d}$ is the same transf

$\Rightarrow$ 2D conformal group

$$\boxed{SL(2,C)/Z_2 \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } ad - bc = 1.}$$

check: $SL(2, C)$

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c \; , & a'b + db' \\ c'a + cd' \; , & c'b + dd' \end{pmatrix}$$

$$z \to z' = \frac{az+b}{cz+d} \quad \Rightarrow z'' = \frac{a'z'+b'}{c'z'+d'} = \frac{(a'a+b'c)z + (a'b+db')}{(c'a+cd')z + (c'b+dd')}$$

$$\frac{a'(az+b) + b'(cz+d)}{c'(az+b) + d'(cz+d)}$$

4 complex numbers $-$ 1 complex constriant $=$ 3 complex

or 6 real degree of free-dom.