3 Coordinate Bethe ansatz - spin ½ fermion

$$H = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i^2} + 2c\sum_{i<j}\delta(x_i - x_j)$$

for N particles, there are N! sectors corresponding to different spatial arrangement of particles. In each sector, the wavefunction is a super position of N! plane waves.

where $P = Q = \begin{pmatrix}1&2\\1&2\end{pmatrix}$, $\bar{P} = \bar{Q} = \begin{pmatrix}1&2\\2&1\end{pmatrix}$

① Let us consider 2- particles

$$\psi_{\sigma_1\sigma_2}(x_1, x_2) = \theta(x_1 < x_2)\left[A_{\sigma_1\sigma_2}(Q, P)e^{i(k_1x_1 + k_2x_2)} + A_{\sigma_1\sigma_2}(Q, \bar{P})e^{i(k_2x_1 + k_1x_2)}\right]$$
$$+ \theta(x_2 < x_1)\left[A_{\sigma_1\sigma_2}(\bar{Q}, P)e^{i(k_1x_2 + k_2x_1)} + A_{\sigma_1\sigma_2}(\bar{Q}, \bar{P})e^{i(k_2x_2 + k_1x_1)}\right]$$

Fermi statistis requires that $x_1\sigma_1 \leftrightarrow x_2\sigma_2$, $\psi_{\sigma_1\sigma_2}(x_1, x_2) = -\psi_{\sigma_2\sigma_1}(x_2, x_1)$

$$\boxed{A_{\sigma_1\sigma_2}(Q, P) = -A_{\sigma_2\sigma_1}(\bar{Q}, P) \quad \text{and} \quad A_{\sigma_1\sigma_2}(Q, \bar{P}) = -A_{\sigma_2\sigma_1}(\bar{Q}, \bar{P})}$$

what's the meaning of $A_{\sigma_1\sigma_2}$, say if for a singlet state $|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2$

it corresponds to $A_{11} = 0$, $A_{1-1} = 1$, $A_{-1,1} = -1$, $A_{-1-1} = 0$.

it satisfies $A_{\sigma_1\sigma_2} = -A_{\sigma_2\sigma_1}$, i.e. $|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2 =$

$$= -\left(|\uparrow\rangle_2|\downarrow\rangle_1 - |\uparrow\rangle_1|\downarrow\rangle_2\right)$$

$A_{\sigma_1\sigma_2}$ the first index always refers to the spin of particle 1, second index refers to the spin of particle 2.

We need to determine $A_{\sigma_1\sigma_2}(Q,P)$ and $A_{\sigma_1\sigma_2}(Q,\bar{P})$. First consider 2- particles. $y = x_2 - x_1$, $X = \frac{x_1+x_2}{2}$, $K = k_1 + k_2$,

$$\psi_{\sigma_1\sigma_2}(x_1,x_2) = \begin{cases} e^{iKX}\left[A_{\sigma_1\sigma_2}(Q,P)\, e^{i(k_2-k_1)y/2} + A_{\sigma_1\sigma_2}(Q\bar{P})\, e^{-i(k_2-k_1)y/2}\right. & (y>0) \\ e^{iKX}\left[A_{\sigma_1\sigma_2}(\bar{Q},P)\, e^{-i(k_2-k_1)y/2} + A_{\sigma_1\sigma_2}(\bar{Q}\bar{P})\, e^{i(k_2-k_1)y/2}\right. & (y<0) \end{cases}$$

$\psi(y=0)$ is continuous $\Rightarrow$ $\underline{A_{\sigma_1\sigma_2}(Q,P) + A_{\sigma_1\sigma_2}(Q,\bar{P}) = A_{\sigma_1\sigma_2}(\bar{Q}P) + A_{\sigma_1\sigma_2}(\bar{Q}\bar{P})}$.

$$\left.\frac{\partial\psi}{\partial y}\right|_{0^+} - \left.\frac{\partial\psi}{\partial y}\right|_{0^-} = c\,\psi\Big|_0$$

$$\Rightarrow \frac{i}{2}(k_2-k_1)\left[A_{\sigma_1\sigma_2}(Q,P) - A_{\sigma_1\sigma_2}(Q\bar{P}) + A_{\sigma_1\sigma_2}(\bar{Q},P) - A_{\sigma_1\sigma_2}(\bar{Q}\bar{P})\right]$$

$$= c\left[A_{\sigma_1\sigma_2}(Q,P) + A_{\sigma_1\sigma_2}(Q,\bar{P})\right]$$

define the exchange operater $\hat{P}_{Q_1Q_2}$ on the "Q".

$$\hat{P}_{Q_1Q_2}(Q_1 Q_2) = (Q_2 Q_1)$$

$$A_{\sigma_1\sigma_2}(\bar{Q},\bar{P}) = \hat{P}_{Q_1Q_2} A_{\sigma_1\sigma_2}(Q,\bar{P})$$

$$\Rightarrow \frac{i}{2}(k_2-k_1)\left[(1+\hat{P}_{Q_1Q_2})(A_{\sigma_1\sigma_2}(Q,P) - A_{\sigma_1\sigma_2}(Q,\bar{P}))\right] = c\left[A_{\sigma_1\sigma_2}(QP) + A_{\sigma_1\sigma_2}(Q\bar{P})\right]$$

$$\cancel{A_{\sigma_1\sigma_2}(Q,P) = \frac{(k_1-k_2)P_{Q_1Q_2} + ic}{k_1-k_2-ic} A_{\sigma_1\sigma_2}(Q,P)}$$

$$\left[\frac{i}{2}(k_2-k_1)(1+\hat{P}_{Q_1Q_2}) - c\right] A_{\sigma_1\sigma_2}(Q,P) = \left[\frac{i}{2}(k_2-k_1)(1+\hat{P}_{Q_1Q_2}) A_{\sigma_1\sigma_2}(Q\bar{P}) + c\right]$$

$$\left[\frac{i}{2}(k_2-k_1)\hat{P}_{Q_1Q_2} + c - \frac{i}{2}(k_2-k_1)\right]\left[\frac{i}{2}(k_2-k_1)\hat{P}_{Q_1Q_2} - c + \frac{i}{2}(k_2-k_1)\right] A_{\sigma_1\sigma_2}(Q,P)$$

$$= \left[\frac{i}{2}(k_2-k_1)(\hat{P}_{Q_1Q_2}-1) + c\right]\left[\frac{i}{2}(k_2-k_1)(\hat{P}_{Q_1Q_2}+1) + c\right] A_{\sigma_1\sigma_2}[Q,\bar{P}]$$

$$\left[-c^2 + ic(k_2-k_1)\right] A_{\sigma_1\sigma_2}(Q,P) = \left[ic(k_2-k_1)\hat{P}_{Q_1Q_2} + c^2\right] A_{\sigma_1\sigma_2}[Q,\bar{P}]$$

$$\Rightarrow \quad A_{\sigma_1\sigma_2}[Q,P] = \frac{(k_1-k_2)\hat{P}_{Q_1Q_2} + ic}{(k_1-k_2) - ic} A_{\sigma_1\sigma_2}[Q,\bar{P}]$$

define $$Y^{12}_{P_2P_1} = \frac{(k_{P_1}-k_{P_2})\hat{P}_{Q_1Q_2} + ic}{k_{P_1}-k_{P_2} - ic} \Rightarrow A_{\sigma_1\sigma_2}(Q,P] = Y^{12}_{P_2P_1} A_{\sigma_1\sigma_2}[Q,\bar{P}]$$

* For the general situation, for N-particles.

$$\psi_{\sigma_1\sigma_2\cdots\sigma_N}(x_1 x_2 \cdots x_N) = \sum_Q \sum_P \Theta(x_{Q_1} < x_{Q_2} < \cdots < x_{Q_N}) A_{\sigma_1\sigma_2\cdots\sigma_N}(Q,P) \cdot \exp[ik_{P_1}x_{Q_1} + ik_{P_2}x_{Q_2} + \cdots ik_{P_N}x_{Q_N}]$$

$Q = [Q_1, Q_2, \cdots Q_N]$ ~~$Q' = [Q'_1$~~

$P = [P_1, P_2 \cdots P_N]$ ~~where $\xi = Q_a,\ \eta = Q_b,$~~

~~Fermi statistics~~ $\cancel{\Rightarrow A_{\cdots\sigma_\xi\cdots\sigma_\eta}(Q,P) = -A_{\cdots\sigma_\eta\cdots\sigma_\xi\cdots}(Q',P)}$

define $Q' = (Q'1, Q'2, \cdots Q'a, Q'b \cdots\cdots)$ i.e. $Q'a = Qb \overset{=\eta}{,}\ Q'b = Qa = \xi$

$= (Q1\ Q2, \cdots Qb, Qa \cdots\cdots)$ $Q'_j = Q_j$

in the region of Q and Q', Ψ equals

$$\theta(\cdots < x_{Qa} < x_{Qb} < \cdots)\, A_{\cdots\sigma_\xi\cdots\sigma_\eta\cdots}(Q,P)\, e^{i\sum k_{P_j} x_{Q_j}}$$

$$+\, \theta(\cdots < x_{Qb} < x_{Qa} < \cdots)\, A_{\cdots\sigma_\xi\cdots\sigma_\eta\cdots}(Q',P)\, e^{i\sum k_{P_j} x_{Q'_j}}$$

Fermi statistics.

$$\equiv -\,\theta(\cdots < x_{Qb} < x_{Qa} \cdots)\, A_{\cdots\sigma_\eta\cdots\sigma_\xi}(Q,P)\, e^{i\sum k_{P_j} x_{Q'_j}}$$

$$-\,\theta(\cdots < x_{Qa} < x_{Qb} \cdots)\, A_{\cdots\sigma_\eta\cdots\sigma_\xi}(Q',P)\, e^{i\sum k_{P_j} x_{Q_j}}$$

$$\Rightarrow \boxed{A_{\cdots\sigma_\xi\cdots\sigma_\eta\cdots}(Q,P) = -\,A_{\cdots\sigma_\eta\cdots\sigma_\xi}(Q',P)}$$

For the general case $b = a+1$ (nearest neighbour)

For $P = (\cdots \boxed{P_a\ P_b} \cdots)$ $P' = (\cdots P_b\ P_a \cdots)$

$Q = (\cdots Q_a Q_b \cdots)$ $Q' = (\cdots Q_b Q_a \cdots)$

the continuity equation reads

$$\frac{i}{2}(k_{P_b} - k_{P_a})\big(A_{\sigma_1\cdots\sigma_N}(Q,P) - A_{\sigma_1\cdots\sigma_N}(Q,P') + A_{\sigma_1\cdots\sigma_N}(Q',P)$$

$$- A_{\sigma_1\cdots\sigma_N}(Q',P')\big) = c\,\big(A_{\sigma_1\cdots\sigma_N}(Q,P) + A_{\sigma_1\cdots\sigma_N}(Q,P')\big)$$

define $$y^{ab}_{P_b P_a} = \frac{(k_{P_a} - k_{P_b})\, P_{Q_a Q_b} + ic}{(k_{P_a} - k_{P_b}) - ic}$$

$$\Rightarrow \boxed{A_{\sigma_1\cdots\sigma_N}(Q,P) = y^{ab}_{P_b P_b}\, A_{\sigma_1\cdots\sigma_N}(Q,P')}$$

in other words $\Rightarrow$

$$A_{\sigma_1 \cdots \sigma_N}(Q, P) = \frac{k_{p_a} - k_{p_b}}{k_{p_a} - k_{p_b} - ic} A_{\sigma_1 \cdots \sigma_N}(Q', P') + \frac{ic}{k_{p_a} - k_{p_b} - ic} A_{\sigma_1 \cdots \sigma_N}(Q, P')$$

$$A_{\cdots \sigma_{Qa} \cdots \sigma_{Q_b} \cdots}(Q\ P') = - A_{\cdots \sigma_{Qb} \cdots \sigma_{Qa} \cdots}(Q', P')$$

define spin-exchange operator $\hat{P}_{\sigma_1 \sigma_2} = \frac{1}{2}(1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) = \begin{cases} 1 & \text{triplet} \\ -1 & \text{singlet} \end{cases}$

$$\hat{P}_{\sigma_1 \sigma_2} \begin{array}{l} |\uparrow_1 \uparrow_2\rangle = |\uparrow_1 \uparrow_2\rangle \\ |\uparrow_1 \downarrow_2\rangle = |\downarrow_1 \uparrow_2\rangle \\ |\downarrow_1 \uparrow_2\rangle = |\uparrow_1 \downarrow_2\rangle \\ |\downarrow_1 \downarrow_2\rangle = |\downarrow_1 \downarrow_2\rangle \end{array}$$

$$\Rightarrow A_{\cdots \sigma_{Qa} \cdots \sigma_{Qb} \cdots}(Q, P') = - P_{\sigma_{Qa} \sigma_{Qb}} A_{\cdots \sigma_{Qa} \sigma_{Qb}}(Q', P')$$

$$\Rightarrow \boxed{A_{\sigma_1 \cdots \sigma_N}(Q, P) = \frac{-k_{p_a} + k_{p_b} + ic P_{\sigma_{Qa} \sigma_{Qb}}}{-k_{p_a} + k_{p_b} + ic} A_{\sigma_1 \cdots \sigma_N}(Q', P') = S(k_{p_b} - k_{p_a}) A_{\sigma_1 \cdots \sigma_N}(Q', P')}$$

where $S(k_{p_b} - k_{p_a}) = \dfrac{k_{p_b} - k_{p_a} + ic P_{\sigma_{Qa} \sigma_{Qb}}}{k_{p_b} - k_{p_a} + ic}$

§ Yang - Baxter Eq.

$$A_{\sigma_1\cdots\sigma_N}(Q, \cdots \overset{\downarrow P}{\underset{\substack{\uparrow \\ a\text{th}}}{i}\,\underset{\substack{\uparrow \\ b\text{th}}}{j}} \cdots) = Y^{ab}_{ji}\, A_{\sigma_1\cdots\sigma_N}(Q, \cdots j i \cdots)$$

adjacent

the positions

$$Y^{ab}_{ji} = \frac{(k_i - k_j)\, P_{Q_aQ_b} + ic}{(k_i - k_j) - ic}$$

the numbers in that position

using Y^{ab}_{ij} can chang $A_{\sigma_1\cdots\sigma_N}(Q, P=(12\cdots N))$ to arbitany $A_{\sigma_1\cdots\sigma_N}(Q,P)$, but the methods can be more than one. We need the consistency conditions:

① $Y^{ab}_{ij}\, Y^{ab}_{ji} = \underline{1}$ ← check $\dfrac{(k_j - k_i)P_{Q_aQ_b} + ic}{(k_j - k_i) - ic} \cdot \dfrac{(k_i - k_j)P_{Q_aQ_b} + ic}{(k_i - k_j) - ic}$

$$= \frac{-(k_i - k_j)^2 - c^2}{-(k_j - k_i)^2 - c^2} = 1$$

② $Y^{ab}_{ij}\, Y^{cd}_{kl} = Y^{cd}_{kl}\, Y^{ab}_{ij}$ when $(a\,b)$, (c,d) have no common elements.

$$\boxed{③\quad Y^{ab}_{jk}\, Y^{bc}_{ik}\, Y^{ab}_{ij} = Y^{bc}_{ij}\, Y^{ab}_{ik}\, Y^{bc}_{jk}}$$

$\boxed{\text{brutal force check}}$

$$y^{ab}_{jk}\, y^{bc}_{ik}\, y^{ab}_{ij} = \frac{(k_k-k_j)P_{Q_aQ_b}+ic}{(k_k-k_j)-ic}\;\frac{(k_k-k_i)P_{Q_bQ_c}+ic}{(k_k-k_i)-ic}\;\frac{(k_j-k_i)P_{Q_aQ_b}+ic}{(k_j-k_i)-ic}$$

$$y^{bc}_{ij}\, y^{ab}_{ik}\, y^{bc}_{jk} = \frac{(k_j-k_i)P_{Q_bQ_c}+ic}{(k_j-k_i)-ic}\;\frac{(k_k-k_i)P_{Q_aQ_b}+ic}{(k_k-k_i)-ic}\;\frac{(k_k-k_j)P_{Q_bQ_c}+ic}{(k_k-k_j)-ic}$$

denominators are the same, numerators:

$$= P_{Q_aQ_b}P_{Q_bQ_c}P_{Q_aQ_b} = \begin{pmatrix}a&b&c\\b&a&c\end{pmatrix}\begin{pmatrix}a&b&c\\a&c&b\end{pmatrix}\begin{pmatrix}a&b&c\\b&a&c\end{pmatrix} = \begin{pmatrix}c&a&b\\c&b&a\end{pmatrix}\begin{pmatrix}b&a&c\\c&a&b\end{pmatrix}\begin{pmatrix}a&b&c\\b&a&c\end{pmatrix} = \begin{pmatrix}a&b&c\\c&b&a\end{pmatrix}$$

$P_{Q_aQ_c}$ $\parallel$

$$= P_{Q_bQ_c}P_{Q_aQ_b}P_{Q_bQ_c} = \begin{pmatrix}a&b&c\\a&c&b\end{pmatrix}\begin{pmatrix}a&b&c\\b&a&c\end{pmatrix}\begin{pmatrix}a&b&c\\a&c&b\end{pmatrix} = \begin{pmatrix}b&c&a\\c&b&a\end{pmatrix}\begin{pmatrix}a&c&b\\b&c&a\end{pmatrix}\begin{pmatrix}a&b&c\\a&c&b\end{pmatrix} = \begin{pmatrix}a&b&c\\c&b&a\end{pmatrix}$$

qudratic terms inovalving $P\,P$ are identical.

linear terms $\left\{P_{Q_aQ_b}(k_k-k_j+k_j-k_i) + (k_k-k_i)P_{Q_bQ_c}\right\}(ic)^2$

$$= \left\{P_{Q_aQ_b}(k_i-k_i) + (k_j-k_i+k_k-k_j)P_{Q_bQ_c}\right\}(ic)^2$$

example 3-particle

$$\begin{tikzcd}
 & A_{\sigma\sigma\sigma}(Q,213) \arrow[r, "y^{23}_{13}"'] & A_{\sigma\sigma\sigma}(Q,231) \arrow[dr, "y^{12}_{23}"'] & \\
A_{\sigma_1\sigma_2\sigma_3}(Q,123) \arrow[ur, "y^{12}_{12}"'] \arrow[dr, "y^{23}_{23}"'] & & & A_{\sigma\sigma\sigma}(Q,321) \\
 & A_{\sigma\sigma\sigma}(Q,132) \arrow[r, "y^{12}_{13}"'] & A_{\sigma\sigma\sigma}(Q,312) \arrow[ur, "y^{23}_{12}"'] &
\end{tikzcd}$$

$$y^{12}_{23}\,y^{23}_{13}\,y^{12}_{12} \longleftrightarrow y^{23}_{12}\,y^{12}_{13}\,y^{23}_{23}$$

or

index of position

$$A_{\sigma\sigma\cdots\sigma}(Q, \cdots \overset{a}{i}\,\overset{b}{j}\,\overset{c}{k} \cdots)$$

index of momentum

$$\begin{tikzcd}[column sep=small]
 & A_{\sigma\sigma\cdots\sigma}(Q, \cdots i j k \cdots) \arrow[dl, "Y^{ab}_{ij}"'] \arrow[dr, "Y^{bc}_{jk}"] & \\
A_{\sigma\sigma\sigma\cdots}(Q, \cdots j i k \cdots) \arrow[d, "Y^{bc}_{ik}"] & & A_{\sigma\cdots\sigma}(Q, \cdots i k j \cdots) \arrow[d, "Y^{ab}_{ik}"] \\
A_{\sigma\cdots\sigma}(Q, j k i) \arrow[dr, "Y^{ab}_{jk}"] & & A_{\sigma\cdots\sigma}(Q, \cdots k i j \cdots) \arrow[dl, "Y^{bc}_{ij}"'] \\
 & A_{\sigma\cdots\sigma}(Q, \cdots k j i \cdots) &
\end{tikzcd}$$

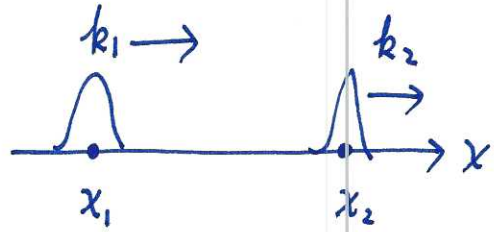
$$Y^{ab}_{jk}\, Y^{bc}_{ik}\, Y^{ab}_{ij} = Y^{bc}_{ij}\, Y^{ab}_{ik}\, Y^{bc}_{jk}$$

§ Scattering matrix,

$$\psi(x_1\sigma_1; x_2\sigma_2) = \Theta(x_1<x_2)\left[A_{\sigma_1\sigma_2}(12;12)\,e^{ik_1x_1+ik_2x_2} + A_{\sigma_1\sigma_2}(12;21)\,e^{ik_2x_1+ik_1x_2}\right]$$
$$+\,\Theta(x_2<x_1)\left[A_{\sigma_1\sigma_2}(21;12)\,e^{ik_1x_2+ik_2x_1} + A_{\sigma_1\sigma_2}(21;21)\,e^{ik_2x_2+ik_1x_1}\right]$$

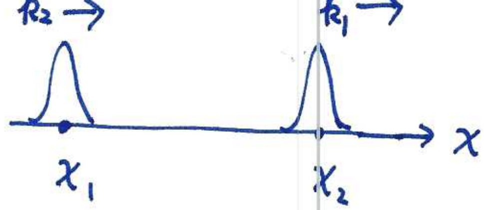
$A_{\sigma_1\sigma_2}(12;12)\,e^{ik_1x_1+ik_2x_2}$

incoming wave 1



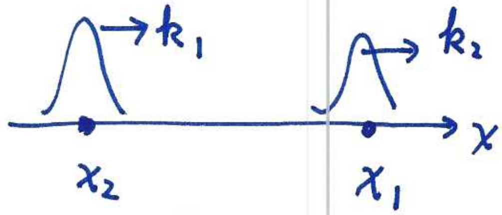
$A_{\sigma_1\sigma_2}(12;21)\,e^{ik_2x_1+ik_1x_2}$

out going wave 2



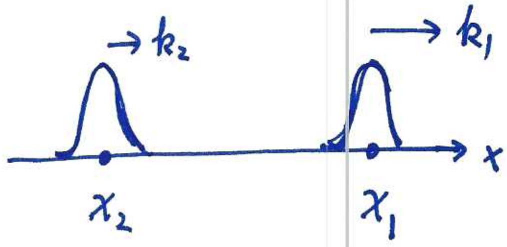
$A_{\sigma_1\sigma_2}(21;12)\,e^{ik_1x_2+k_2x_1}$

in coming wave 2



$A_{\sigma_1\sigma_2}(21;21)\,e^{ik_2x_2+ik_1x_1}$

out going wave 1



$$A_{\sigma_1\sigma_2}(12;12) = \underset{\sigma_1\sigma_2,\,\sigma_1'\sigma_2'}{S(k_2-k_1)}\,A_{\sigma_1'\sigma_2'}(21;21)$$

$$A_{\sigma_1\sigma_2}(21;12) = \underset{\sigma_1\sigma_2,\,\sigma_1'\sigma_2'}{S(k_2-k_1)}\,A_{\sigma_1'\sigma_2'}(12,21)$$

Periodical boundary condition

$$\psi(\underset{\overset{\|}{0}}{x_1}\sigma_1, x_2\sigma_2, \cdots x_N\sigma_N) = \psi(x_1 = L, \sigma_1, x_2\sigma_2, \cdots x_N\sigma_N)$$

$$\Rightarrow \sum_Q \sum_P \Theta(x_{Q1} = x_1 = 0 < x_{Q2} < \cdots < x_{QN}) A_{\sigma_1\cdots\sigma_N}(Q,P) e^{ik_{P_1}0 + ik_{P_2}x_{Q2} + \cdots ik_{P_N}x_{QN}}$$

$$= \sum_Q \sum_P \Theta(x_{Q_2} < \cdots < x_{QN} < x_{Q_1} = L) A_{\sigma_1\cdots\sigma_N}(Q_2\cdots Q_N, 1, P) e^{ik_{P_1}x_{Q2} + ik_{P_{N-1}}x_{QN} + ik_{P_N}L}$$

change variable $P \to (P_2, P_3 \cdots P_N, P_1)$

$$\sum_Q \sum_P \Theta(x_{Q_2} < \cdots < x_{QN} < x_{Q_1} = L) A_{\sigma_1\cdots\sigma_N}(Q_2, \cdots Q_N 1; P_2 \cdots P_N P_1) e^{ik_{P_2}x_{Q_2} + \cdots} \cdot e^{ik_{P_1} \cdot L}$$

$$\Rightarrow A_{\sigma_1\cdots\sigma_N}(Q_1\cdots Q_N, P_1\cdots P_N) = A_{\sigma_1\cdots\sigma_N}(Q_2\cdots Q_N 1; P_2\cdots P_N P_1) e^{ik_{P_1}L}$$

Consider the case of

$$(Q1 \cdots QN) = (P1 \cdots PN) = (j, \overset{Q_2}{\overset{\downarrow}{1}}, \cdots j-1; j+1, \cdots N)$$

$$\Rightarrow A_{\sigma_1\cdots\sigma_N}(\underline{j\,1\cdots j-1, j+1\cdots N}; \underline{j\,1\cdots j-1, j+1\cdots N})$$

$$= e^{ik_jL} A_{\sigma_1\cdots\sigma_N}(\underline{1\cdots j-1, j+1\cdots N, j}; \underline{1\cdots j-1, j+1, \cdots N, j})$$

using $A_{\sigma_1\cdots\sigma_N}(Q,P) = S(k_{P_b} - k_{P_a}) A_{\sigma_1\cdots\sigma_N}(Q', P')$

if $P = (\cdots P_a P_b \cdots)$ $P' = (\cdots P_b P_a \cdots)$ Recall page #

$Q = (\cdots Q_a Q_b \cdots)$ $Q' = (\cdots Q_b Q_a \cdots)$

$$S(k_{P_b} - k_{P_a}) = \frac{k_{P_b} - k_{P_a} + ic \; P_{\sigma_{Q_a}, \sigma_{Q_b}}}{k_{P_b} - k_{P_a} + ic}.$$

define $S_{ij} = \dfrac{k_i - k_j + ic \; P_{\sigma_i \sigma_j}}{k_i - k_j + ic}$ set $P_a = Q_a = i$, $P_b = Q_b = j$

$$\Rightarrow A_{\sigma_1 \cdots \sigma_N}(j\, 1 2 \cdots j-1, j+1 \cdots N;\; j\, 1 2 \cdots j-1, j+1 \cdots N)$$

$$= S_{1j} \, S_{2j} \cdots S_{j-2,j} \, S_{j-1,j} \, A_{\sigma_1 \cdots \sigma_N}(1 2 \cdots N, 1 2, \cdots N)$$

$$A_{\sigma_1 \cdots \sigma_N}(1 \cdots j-1, j+1 \cdots N j;\; 1 2 \cdots j-1, j+1 \cdots N, j)$$

$$= S_{j,N} \cdots S_{j,j+2} \, S_{j,j+1} \, A_{\sigma_1 \cdots \sigma_N}(1 2 \cdots N, 1 2, \cdots N)$$

$$\Rightarrow [S_{j+1,j} \cdots S_{N,j} \, S_{1j} \, S_{2j} \cdots S_{j-1,j}] \, A_{\sigma_1 \cdots \sigma_N}(1 2 \cdots N, 1 2, \cdots N)$$

$$= e^{i k_j L} \, A_{\sigma_1 \cdots \sigma_N}(1 2 \cdots N, 1 2 \cdots N).$$

We need to solve $A_{\sigma_1 \cdots \sigma_N}$ as an eigenvector of spin-operator $\underline{S_{j+1,j} \cdots S_{Nj} \, S_{1j} \, S_{2j} \cdots S_{j-1,j}}$.

once we solve the eigenvalue of k_j, $\Rightarrow E = \sum_{j=1}^{N} k_j^2$

Quantum inverse method:

define an auxilary space C_2 (A) and $\vec{\tau}$ as Pauli matrix in such a space. Define $P^{j,A} = \frac{1}{2}(1 + \vec{\sigma}_j \cdot \vec{\tau})$, and the auxillary S-matrix

$$S^{jA}[u] = \frac{k_j - cu}{k_j - cu + ic} + \frac{ic}{k_j - cu + ic} P^{j,A}$$

The monodromy matrix:

$$T(u) = S^{1A}[u]\, S^{2A}[u] \cdots S^{NA}[u],$$ where the matrix product only acts in the auxillary space A.

$$T_{\underline{\sigma_1\sigma_2\cdots\sigma_N}\, u\,;\, \underline{\sigma_1'\sigma_2'\cdots\sigma_N'}\, v} = S^{1A}_{\sigma_1 u;\sigma_1'\omega_1} S^{2A}_{\sigma_2\omega_1;\sigma_2'\omega_2} S^{3A}_{\sigma_3\omega_2;\sigma_3'\omega_3} \cdots S^{NA}_{\sigma_N\omega_{N-1};\sigma_N' v}$$

if we trace over the auxillary space $\Rightarrow$

$$tr_A[T(u_j)] = \underline{S^{j+1\,j} S^{j+2\,j} \cdots S^{Nj} S^{1j} \cdots S^{j-1\,j}}$$

where $u_j = \frac{k_j}{c}$.

Proof: $$tr_A[T(u_j)] = S^{1A}_{\sigma_1 u_1,\sigma_1' u_2} S^{2A}_{\sigma_2 u_2,\sigma_2' u_3} \cdots S^{jA(u_j)}_{\sigma_j u_j,\sigma_j' u_{j+1}} \cdots S^{N,A}_{\sigma_N u_N,\sigma_N' u_1}$$

$$= S^{j+1,A}_{\sigma_{j+1} u_{j+1};\sigma_{j+1}' u_{j+2}} \cdots S^{N,A}_{\sigma_N u_N,\sigma_N' u_1} \cdot S^{1A}_{\sigma_1 u_1,\sigma_1' u_2} \cdots S^{j-1\,A}_{\sigma_{j-1}u_{j-1},\sigma_{j-1}' u_j}\ S^{jA(u_j)}_{\sigma_j u_j,\sigma_j' u_{j+1}}$$

$$S^{j,A} = P^{j,A} = \frac{1}{2}(1+\vec{\sigma}_j\cdot\vec{\tau})$$

$$\boxed{S^{l,A}(u_j)\ P^{j,A} = P^{j,A}\ S^{l,j}}$$

Check
$$\left[\frac{k_l-k_j}{k_l-k_j+ic} + \frac{ic}{k_l-k_j+ic}P^{l,A}\right]P^{j,A}$$

where
$$P^{l,A}P^{j,A} = \frac{1}{4}(1+\vec{\sigma}_l\cdot\vec{\tau})(1+\vec{\sigma}_j\cdot\vec{\tau})$$

$$= \frac{1}{4}[1+\vec{\sigma}_l\cdot\vec{\tau}+\vec{\sigma}_j\cdot\vec{\tau} + \sigma_{l\alpha}\sigma_{j\beta}\tau_\alpha\tau_\beta)$$

$$= \frac{1}{4}[1+\vec{\sigma}_l\cdot\vec{\tau}+\vec{\sigma}_j\cdot\vec{\tau}+\vec{\sigma}_l\cdot\vec{\sigma}_j + i(\vec{\sigma}_l\times\vec{\sigma}_j)\cdot\vec{\tau}]$$

$$= \frac{1}{4}(1+\vec{\sigma}_j\cdot\vec{\tau})(1+\vec{\sigma}_j\cdot\vec{\sigma}_l) = P^{j,A}\ P^{lj}$$

$$\Rightarrow\quad P^{j,A}\left[\frac{k_l-k_j}{k_l-k_j+ic} + \frac{ic}{k_l-k_j+ic}P^{lj}\right]$$

$$\mathrm{tr}_A[T(u_j)] = \mathrm{tr}_A[S^{j+1,A}\dots S^{N,A}\,S^{1A}\dots S^{j-1,A}\,P^{j,A}]$$

$$= \underbrace{\mathrm{tr}_A[P^{j,A}]}_{=1}\underbrace{S^{j+1,j}\dots S^{N,j}\,S^{1,j}\dots S^{j-1,j}}_{\text{cnst in }A\text{-space}}]$$

$$= \underline{S^{j+1,j}\dots S^{N,j}\,S^{1,j}\dots S^{j-1,j}}$$

Then the Bethe ansatz equation reduces to

$$\left[tr_A\, T\left(\frac{k_j}{c}\right)\right]_{\sigma_1\cdots\sigma_N,\,\sigma_1'\cdots\sigma_N'} A_{\sigma_1'\cdots\sigma_N'}(12\cdots N, 12\cdots N) = e^{ik_jL} A_{\sigma_1\cdots\sigma_N}^{(1\cdots N, 1\cdots N)}$$

we need to diagonalize $tr_A\, T\left(\frac{k_j}{c}\right)$ simultanously for $j = 1, 2, 3, \cdots N$.

i.e. we need to prove $[tr_A\, T[u],\ tr_B\, T[v]] = 0$ for arbitany u, v.

For this purpose, we define R-matrix defined in the $A \otimes B$ space

$$R^{AB}(u) = \frac{i}{u+i} + \frac{u}{u+i} P^{AB}, \quad \text{where } P^{AB} = \frac{1}{2}(1 + \vec{\tau}^A \vec{\tau}^B)$$

$$= \left(\frac{u + iP^{AB}}{u+i}\right) P^{AB}$$

If we write the matrix indices explicitly, $P^{AB}_{uw,u'w'} = \delta_{uw'}\delta_{wu'}$.

Define $S^{jA} \otimes S^{jB}$, the matrix product in the "j"-spin space

but keeps the direct product in A, and B. space.

Now let us define

$R^{AB}(u-v) \cdot S^{jA}(u) \cdot S^{jB}(v)$, the matrix product takes in the same space

and $S^{jA}(v)\, S^{jB}(u)\, R^{AB}(u-v)$

$A \Leftrightarrow A$, $B \Leftrightarrow B$,
$j \Leftrightarrow j$

$$R^{AB}(u-v)\, S^{jA}(u)\, S^{jB}(v) = P^{AB}\left(\frac{u-v+iP^{AB}}{u-v+i}\right)\left(\frac{u_j-u+iP^{j,A}}{u_j-u+i}\right)\left(\frac{u_j-v+iP^{j,B}}{u_j-v+i}\right)$$

$$= P^{AB}\left(\frac{u_j-v+iP^{j,B}}{u_j-v+i}\right)\left(\frac{u_j-u+iP^{j,A}}{u_j-u+i}\right)\left(\frac{u-v+iP^{AB}}{u-v+i}\right)$$

This process is essentially the Yang - Baxer Eq.

$$\begin{pmatrix} \gamma & \beta & \alpha \\ A & B & j \end{pmatrix}$$

$$\uparrow$$

$$P^{AB}\, P^{jA}\, P^{jB} \begin{pmatrix} \alpha & \beta & \gamma \\ A & B & j \end{pmatrix} = P^{AB} P^{jA} \begin{pmatrix} \alpha & \gamma & \beta \\ A & B & j \end{pmatrix} = P^{AB} \begin{pmatrix} \beta & \gamma & \alpha \\ A & B & j \end{pmatrix}$$

$$P^{jB}\, P^{jA}\, P^{AB} \begin{pmatrix} \alpha & \beta & \gamma \\ A & B & j \end{pmatrix} = P^{jB} P^{jA} \begin{pmatrix} \beta & \alpha & \gamma \\ A & B & j \end{pmatrix} = P^{jB} \begin{pmatrix} \gamma & \alpha & \beta \\ A & B & j \end{pmatrix} = \begin{pmatrix} \gamma & \beta & \alpha \\ A & B & j \end{pmatrix}$$

then

$$P^{AB}\, P^{jB} \begin{pmatrix} \alpha & \beta & \gamma \\ A & B & j \end{pmatrix} = P^{AB} \begin{pmatrix} \alpha & \gamma & \beta \\ A & B & j \end{pmatrix} = \begin{pmatrix} \gamma & \alpha & \beta \\ A & B & j \end{pmatrix}$$

$$P^{jA}\, P^{AB} \begin{pmatrix} \alpha & \beta & \gamma \\ A & B & j \end{pmatrix} = P^{jA} \begin{pmatrix} \beta & \alpha & \gamma \\ A & B & j \end{pmatrix} = \begin{pmatrix} \gamma & \alpha & \beta \\ A & B & j \end{pmatrix}$$

$\Rightarrow$ $P^{AB}\, P^{jB} = P^{jA}\, P^{AB}$, similarly $P^{AB}\, P^{j,A} = P^{jB}\, P^{AB}$

$$\Rightarrow R^{AB}(u-v)\, S^{jA}(u)\, S^{jB}(v)$$

$$= \left(\frac{u_j-v+iP^{j,A}}{u_j-v+i}\right)\left(\frac{u_j-u+iP^{j,B}}{u_j-u+i}\right)\left(\frac{u-v+iP^{AB}}{u-v+i}\right) P^{AB}$$

$$\Rightarrow \quad R^{AB}(u-v)\, S^{jA}(u)\, S^{jB}(v) = S^{jA}(v)\, S^{jB}(u)\, R^{AB}(u-v)$$

Similarly

$$\boxed{R^{AB}(u-v)\, T^{A}(u)\, T^{B}(v) = T^{A}(v)\, T^{B}(u)\, R^{AB}(u-v),}$$

where T is matrices in the physical spin space.

define $b(u) = \dfrac{-u}{-u+i}$, $\quad c(u) = \dfrac{i}{-u+i}$

$$S^{ij} = b\left(\frac{k_j}{c} - \frac{k_i}{c}\right) + c\left(\frac{k_j}{c} - \frac{k_i}{c}\right) P^{ij}$$

$$S^{jA}_{(u)} = b\left(u - \frac{k_j}{c}\right) + c\left(u - \frac{k_j}{c}\right) P^{jA}$$

$$R^{AB}(u) = c(u) + b(u)\, P^{AB} = (b(u) + c(u)\, P^{AB})\, P^{AB}$$

$$= \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \uparrow_A \uparrow_B \\ \uparrow_A \downarrow_B \\ \downarrow_A \uparrow_B \\ \downarrow_A \downarrow_B \end{array} \end{array}$$

T is a 2×2 matrix in the A- space, Let us define

$$T(u) = \begin{pmatrix} A(u), & B(u) \\ C(u), & D(u) \end{pmatrix}$$

$$\left[R^{AB}(u-v)\,T^A(u)\,T^B(v)\right]_{uw,\,u'w'} = \left[c(u-v)\,\delta_{uw,\,u''w''} + b(u-v)\,P_{uw,\,u''w''}\right] \overbrace{T^A_{u''u'}(u)\,T^B_{w''w'}(v)}$$

$$= c(u-v)\,T^A_{uu'}(u)\,T^B_{ww'}(v) + b(u-v)\,T^A_{wu'}(u)\,T^B_{uw'}(v)$$

$$\left[T^A(v)\,T^B(u)\,R^{AB}(u-v)\right]_{uw,\,u'w'} = T^A_{uu''}(v)\,T^B_{ww''}(u)\left[c(u-v)\,\delta_{u''w'',\,u'w'} + b(u-v)\,P_{u''w'',\,u'w'}\right]$$

$$= c(u-v)\,T^A_{uu'}(v)\,T^B_{ww'}(u) + b(u-v)\,T^A_{uw'}(v)\,T^B_{wu'}(u)$$

$uw\ u'w'$

$(11,11)$: $\quad T_{11}(u)T_{11}(v) = T_{11}(v)T_{11}(u)$, $\Rightarrow$ $\underline{A(u)A(v) = A(v)A(u)}$,

$(11,12)$ $\quad c(u-v)\,T_{11}(u)\,T_{12}(v) + b(u-v)\,T_{11}(u)\,T_{12}(v)$

$$= c(u-v)\,T_{11}(v)\,T_{12}(u) + b(u-v)\,T_{12}(v)\,T_{11}(u)$$

$$\Rightarrow \quad \underline{A(u)\,B(v) = c(u-v)\,A(v)\,B(u) + b(u-v)\,B(v)\,A(u)}$$

$(11,\cancel{13}\ 21)$ $\quad \cancel{c(u-v)\,T_{11}(u)\,T_{13}(v) + b(u-v)\,T_{11}(u)\,T_{13}(v)}$

$$\cancel{= c(u-v)\,T_{11}(v)\,T_{13}(u) + b(u-v)\,T_{13}(v)\,T_{11}(u)}$$

$$\Rightarrow \quad c(u-v)\,T_{12}(u)\,T_{11}(v) + b(u-v)\,T_{12}(u)\,T_{11}(v)$$

$$= c(u-v)\,T_{12}(v)\,T_{11}(u) + b(u-v)\,T_{11}(v)\,T_{12}(u)$$

$$\underline{B(u)\,A(v) = c(u-v)\,B(v)\,A(u) + b(u-v)\,A(v)\,B(u)}$$

$$\Rightarrow \quad A(v)\,B(u) = \frac{1}{b(u-v)}\,B(u)\,A(v) - \frac{c(u-v)}{b(u-v)}\,B(v)\,A(u)$$

$uw\ u'w'$
$(11,22)$

$$c(u-v)\,T_{12}(u)\,T_{12}(v) + b(u-v)\,T_{12}(u)\,T_{12}(v)$$
$$= c(u-v)\,T_{12}(v)\,T_{12}(u) + b(u-v)\,T_{12}(v)\,T_{12}(u) \Rightarrow \boxed{B(u)B(v) = B(v)B(u)}$$

$uw\ u'w'$
$(12,11)$

$$c(u-v)\,T_{11}(u)\,T_{21}(v) + b(u-v)\,T_{21}(u)\,T_{11}(v)$$
$$= c(u-v)\,T_{11}(v)\,T_{21}(u) + b(u-v)\,T_{11}(v)\,T_{21}(u)$$

$$\underline{c(u-v)\,A(u)\,B(v) + b(u-v)\,B(u)\,A(v) = A(v)\,B(u)}$$

$uw\ u'w'$
$(12,12)$

$$c(u-v)\,T_{11}(u)\,T_{22}(v) + b(u-v)\,T_{21}(u)\,T_{12}(v)$$
$$= c(u-v)\,T_{11}(v)\,T_{22}(u) + b(u-v)\,T_{12}(v)\,T_{21}(u)$$

$$\star\ \underline{c(u-v)\,A(u)\,D(v) + b(u-v)\,C(u)\,B(v) = c(u-v)\,A(v)\,D(u) + b(u-v)\,B(v)\,C(u)}$$

$uw\ u'w'$
$12\ 21$

$$c(u-v)\,T_{12}(u)\,T_{21}(v) + b(u-v)\,T_{22}(u)\,T_{11}(v)$$
$$= c(u-v)\,T_{12}(v)\,T_{21}(u) + b(u-v)\,T_{11}(v)\,T_{22}(u)$$

$$\star\ \underline{c(u-v)\,B(u)\,C(v) + b(u-v)\,D(u)\,A(v)}$$
$$\underline{= c(u-v)\,B(v)\,C(u) + b(u-v)\,A(v)\,D(u)}$$

$uw,\ u'w'$
$12\ \ 22$

$$c(u-v)\,T_{12}(u)\,T_{22}(v) + b(u-v)\,T_{22}(u)\,T_{12}(v)$$
$$= c(u-v)\,T_{12}(v)\,T_{22}(u) + b(u-v)\,T_{12}(v)\,T_{22}(u)$$

$$\underline{c(u-v)\,B(u)\,D(v) + b(u-v)\,D(u)\,B(v)} \Rightarrow \boxed{D(u)B(v) = \frac{B(v)\,D(u)}{b(u-v)} - \frac{c(u-v)}{b(u-v)}\,B(u)D(v)}$$
$$= \cancel{c(u-v)}\ \underline{B(v)\,D(u)}$$

$uw \quad u'w'$

$21 \quad 11$ $\quad c(u-v)\, T_{21}(u) T_{11}(v) + b(u-v)\, T_{11}(u) T_{21}(v)$

$$= c(u-v)\, T_{21}(v) T_{11}(u) + b(u-v)\, T_{21}(v) T_{11}(u)$$

$$\underline{c(u-v)\, \mathbb{C}(u) A(v) + b(u-v)\, A(u)\, \mathbb{C}(v) = \mathbb{C}(v) A(u)}$$

$21 \quad 12$ $\quad c(u-v)\, T_{21}(u) T_{12}(v) + b(u-v)\, T_{11}(u) T_{22}(v)$

$$= c(u-v)\, T_{21}(v) T_{12}(u) + b(u-v)\, T_{22}(v) T_{11}(u)$$

$$\underline{c(u-v)\, \mathbb{C}(u) B(v) + b(u-v)\, A(u) D(v) = c(u-v)\, \mathbb{C}(v) B(u) + b(u-v)\, D(v) A(u)}$$

$21 \quad 21$ $\quad c(u-v)\, T_{22}(u) T_{11}(v) + b(u-v)\, T_{12}(u) T_{21}(v)$

$$= c(u-v)\, T_{22}(v) T_{11}(u) + b(u-v)\, T_{21}(v) T_{12}(u)$$

$$\underline{c(u-v)\, D(u) A(v) + b(u-v)\, B(u) C(v) = c(u-v)\, D(v) A(u) + b(u-v)\, C(v) B(u)}$$

$21 \quad 22$ $\quad c(u-v)\, T_{22}(u) T_{12}(v) + b(u-v)\, T_{12}(u) T_{22}(v)$

$$= c(u-v)\, T_{22}(v) T_{12}(u) + b(u-v)\, T_{22}(v) T_{12}(u)$$

$$\underline{c(u-v)\, D(u) B(v) + b(u-v)\, B(u) D(v) = c(u-v)\, D(v) B(u) + b(u-v)\, D(v) B(u)}$$

$$\boxed{= D(v) B(u)}$$

$22 \quad 11,$ $\quad c(u-v)\, T_{21}(u) T_{21}(v) + b(u-v)\, T_{21}(u) T_{21}(v)$

$$= c(u-v)\, T_{21}(v) T_{21}(u) + b(u-v)\, T_{21}(v) T_{21}(u)$$

$$\cancel{c(u-v)} \boxed{C(u) C(v) = C(v) C(u)}$$

$uw\ u'w'$
$22,12$ $\quad c(u-v)\, T_{21}(u) T_{22}(v) + b(u-v)\, T_{21}(u)\, T_{22}(v)$

$$= c(u-v)\, T_{21}(v)\, T_{22}(u) + b(u-v)\, T_{22}(v)\, T_{21}(u)$$

$$\Rightarrow \underline{C(u)\, D(v) = c(u-v)\, C(v)\, D(u) + b(u-v)\, D(v)\, C(u)}$$

$uw\ u'w'$
$22\ 21$ $\quad c(u-v)\, T_{22}(u)\, T_{21}(v) + b(u-v)\, T_{22}(\cancel{u})\, T_{21}(\cancel{v})$

$$= c(u-v)\, T_{22}(v) T_{21}(u) + b(u-v)\, T_{21}(v)\, T_{22}(u)$$

$$\underline{D(u)\, C(v) = c(u-v)\, D(v)\, C(u) + b(u-v)\, C(v) D(u)}$$

$22\ 22$ $\quad \Rightarrow \quad \underline{D(u)\, D(v) = D(v)\, D(u)}$

I still <u>can not</u> get $[tr_A T(u),\ tr_A T(v)] = [tr_A T(v),\ tr_A T(u)]$

$$\Leftrightarrow [A(u) + B(u)][A(v) + D(v)] = (A(v) + D(v))(A(u) + D(u))$$

we have $[A(u)\ A(v)] = 0$, $[D(u), D(v)] = 0$

but $\underline{A(u) D(v) + D(u) A(v) = A(v) D(u) + D(v) A(u)}$?

In $H_1 \otimes H_2 \cdots \otimes H_n$, define the vacum state $|0\rangle = |\uparrow \cdots \uparrow \cdots\rangle$

$$S^{jA}(u) = b(u-u_j) + c(u-u_j)\, P^{j,A} = b(u-u_j) + c(u-u_j)\, \frac{1}{2}(1+\vec{\sigma}_j \cdot \vec{\tau}_A)$$

$$= b(u-u_j) + \frac{c(u-u_j)(1+\sigma_j^z)}{2}\, \frac{\tau_A^z+1}{2} + \frac{c(u-u_j)(1-\sigma_j^z)}{2}\, \frac{1-\tau_A^z}{2}$$

$$+ \frac{c(u-u_j)}{2}\,[2\sigma_+]\, \frac{\tau_x - i\tau_y}{2} + \frac{c(u-u_j)\, 2\sigma_-}{2}\, \frac{\tau_x + i\tau_y}{2}$$

$$= b(u-u_j) + \begin{bmatrix} \frac{c(u-u_j)}{2}(1+\sigma_j^z) & c(u-u_j)\,\sigma_- \\ c(u-u_j)\,\sigma_+ & \frac{c(u-u_j)}{2}(1-\sigma_j^z) \end{bmatrix}$$

acting on $|0\rangle$

$$\Rightarrow S^{jA}(u)\,|0\rangle = \begin{bmatrix} 1 \, , & c(u-u_j)\,\sigma_- \\ \underbrace{c(u-u_j)\,\sigma_+}_{0}, & b(u-u_j) \end{bmatrix} |0\rangle$$

$$T(u)\,|0\rangle = \begin{pmatrix} 1 & B(u) \\ 0 & \prod_{j=1}^{N} b(u-u_j) \end{pmatrix} |0\rangle$$

each element is an operator acting in the spin space.

i.e.
$$\boxed{\begin{aligned} A(u)\,|0\rangle &= |0\rangle \\ D(u)\,|0\rangle &= \prod_{j=1}^{N} b(u-u_j)\,|0\rangle \end{aligned}}$$

$|0\rangle$ is the eigen state of $tr_A(T(u)) = A(u) + D(u)$ with eigenvalue $\prod_{j=1}^{N} b(u-u_j)$.

$$C(u)\,|0\rangle = 0$$

$$B(u)\,|0\rangle = \sum_j \cdots |\uparrow\uparrow\cdots\underset{j\text{-th}}{\downarrow}\uparrow\uparrow\cdots\rangle$$

to get all the other eigen states of $tr_A(T(u))$, we repeatedly apply the flipping opearator B. on the "femo" state

$B(v_1)\cdots B(v_M)|0\rangle$ is $N-M$ spin up, M spin down,

we will prove that it is an eigenstate. of $tr\, T(u)$.

Form $B(u)B(v) - B(v)B(u) = 0$

$$D(u)B(v) = \frac{1}{b(u-v)} B(v)D(u) - \frac{c(u-v)}{b(u-v)} B(u)D(v)$$

$$A(u)B(v) = \frac{1}{b(v-u)} B(v)A(u) - \frac{c(v-u)}{b(v-u)} B(u)A(v)$$

using these relation

$$\left[A(u) + D(u)\right] B(v_1)\cdots B(v_M)\,|0\rangle$$

$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)$

$$= \left\{ \frac{1}{\prod_{\alpha=1}^{M} b(v_\alpha - u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{\prod_{\alpha=1}^{M} b(u-v_\alpha)} \right\} B(v_1)\cdots B(v_M)\,|0\rangle$$

$+ \cdots$ (unwanted terms)

If unwanted terms are zero, then

$B(v_1) \cdots B(v_M)|0\rangle$ is $tr_A T(u)$'s eigenstate with eigenvalue of

$$\prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha - u)} + \frac{\prod_{j=1}^{N} b(u-u_j)}{\prod_{\alpha=1}^{M} b(u-v_\alpha)}$$

$b(0)=0.$

$tr_A T_N(u_j)$'s eigenvalue is $\prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha - u_j)}$

Thus $$\boxed{e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{1}{b(v_\alpha - u_j)},}$$ but we need to decide suitable v_α to vanish unwanted terms.

Now let us collect the next order of unwanted terms.

$$-\frac{c(v_1-u)}{b(v_1-u)}\left[\frac{1}{\prod_{\alpha=2}^{M} b(v_\alpha-u)} - \frac{\prod_{j=1}^{N} b(v_1-u_j)}{\prod_{\alpha=2}^{M} b(v_1-v_\alpha)}\right] B(u)B(v_2)\cdots B(v_M)|0\rangle$$

if we set $\prod_{j=1}^{N} b(v_1-u_j) = \prod_{\alpha=2}^{M} \frac{b(v_1-v_\alpha)}{b(v_\alpha-v_1)}$, the second unwanted terms $\to 0$.

Similarly if we impose

$$\boxed{\prod_{j=1}^{N} b(v_k-u_j) = \prod_{\substack{\alpha=1 \\ \alpha\neq k}}^{M} \frac{b(v_k-v_\alpha)}{b(v_\alpha-v_k)}}$$

$\Rightarrow$ all the unwanted terms go away.

Plug in $u_j = \dfrac{k_j}{c}$, $v_\alpha = \dfrac{\Lambda_\alpha}{c} + \dfrac{i}{2}$, $b(u) = \dfrac{-u}{-u+i}$

$$\Rightarrow \quad b(v_\alpha - u_j) = \frac{-\frac{\Lambda_\alpha}{c} + \frac{k_j}{c} - \frac{i}{2}}{-\frac{\Lambda_\alpha}{c} + \frac{k_j}{c} + \frac{i}{2}} = \frac{k_j - \Lambda_\alpha - \frac{ic}{2}}{k_j - \Lambda_\alpha + \frac{ic}{2}}$$

$$b(v_\alpha - v_\kappa) = \frac{\Lambda_\kappa - \Lambda_\alpha}{\Lambda_\kappa - \Lambda_\alpha + ic},$$

$$b(v_\kappa - v_\alpha) = \frac{\Lambda_\alpha - \Lambda_\kappa}{\Lambda_\alpha - \Lambda_\kappa + ic}$$

$$\Rightarrow \quad \boxed{e^{ik_jL} = \prod_{\alpha=1}^{M} \frac{k_j - \Lambda_\alpha + \frac{ic}{2}}{k_j - \Lambda_\alpha - \frac{ic}{2}}} \leftarrow \text{for each "} k_j \text{"} \; j = 1, \cdots N$$

$$\prod_{j=1}^{N} \frac{(k_j - \Lambda_\kappa) - \frac{ic}{2}}{k_j - \Lambda_\kappa + \frac{ic}{2}} = \prod_{\alpha=1, \alpha\neq\kappa}^{M} \frac{\Lambda_\alpha - \Lambda_\kappa - ic}{\Lambda_\alpha - \Lambda_\kappa + ic}$$

or rewrite

$$\boxed{\prod_{j=1}^{N} \frac{k_j - \Lambda_\alpha + \frac{ic}{2}}{k_j - \Lambda_\alpha - \frac{ic}{2}} = - \prod_{\beta=1}^{M} \frac{\Lambda_\beta - \Lambda_\alpha + ic}{\Lambda_\beta - \Lambda_\alpha - ic} \quad \text{for } \alpha = 1, \cdots M}$$

$$\Rightarrow k_j L = 2\pi \otimes \text{integer} + \sum_{\alpha=1}^{M} 2\tan^{-1} \frac{c}{2(k_j - \Lambda_\alpha)}$$

$$= 2\pi \otimes \text{integer} + \sum_{\alpha=1}^{M} \pi - 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c}$$

$$\boxed{\begin{array}{l} k_j L = 2\pi I_j - 2\sum_{\alpha=1}^{M} \tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} \qquad (j = 1, 2, \cdots N) \\ I_j = \begin{cases} \text{integer} & \text{if } M \text{ is even,} \\ \text{half integer} & \text{if } M \text{ is odd.} \end{cases} \end{array}}$$

$$\sum_{j=1}^{N} 2\tan^{-1} \frac{c}{2(k_j - \Lambda_\alpha)} = 2\pi \otimes \text{integer} + \pi + \sum_{\beta=1}^{M} 2\tan^{-1} \frac{c}{\Lambda_\beta - \Lambda_\alpha}$$

$$\sum_{j=1}^{N} \pi - \sum_{j=1}^{N} 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = 2\pi \otimes (\text{half intger}) + \sum_{\beta=1}^{M} \pi - \sum_{\beta=1}^{M} 2\tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$-\sum_{j=1}^{N} 2\tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = 2\pi \otimes \left(\text{half integer} + \frac{N-M}{2}\right) - \sum_{\beta=1}^{M} 2\tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$\Rightarrow \boxed{\begin{array}{l} 2\sum_{j=1}^{N} \tan^{-1} \frac{2(k_j - \Lambda_\alpha)}{c} = 2\pi J_\alpha + 2\sum_{\beta=1}^{M} \tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}, \quad \alpha = 1, \cdots, M \\ \text{where } J_\alpha = \begin{cases} \text{integer} & \text{if } N-M \text{ is odd} \\ \text{half integer} & \text{if } N-M \text{ is even.} \end{cases} \end{array}}$$

Next we prove :

$$S^{+} B(v_1) \cdots B(v_m) |0\rangle,$$

$$S^{3} B(v_1) \cdots B(v_m) |0\rangle = \frac{N-2m}{2} B(v_1) \cdots B(v_m) |0\rangle,$$

where $S^{\pm} = S^1 \pm i S^2$, i.e. $B(v_1) \cdots B(v_m)$ is $S_{tot} = \frac{N-2m}{2}$ state.

Proof: $[S^{j,A}(u), S_j^n] = [b(u-u_j) + c(u-u_j) \frac{1+\vec{\tau}\cdot\vec{\sigma}_j}{2}, \frac{\sigma_j^n}{2}]$ ($n=1,2,3$ spin orientation)

$$= \frac{1}{4} c(u-u_j) \quad \tau^m [\sigma_j^m \sigma_j^n] \leftarrow 2i\epsilon^{mnl} \tau^m \sigma^l$$

$$\parallel$$

$$= -\frac{1}{4} c(u-u_j) \quad \sigma_j^l [\tau_j^l \tau^n] \quad 2i\epsilon^{lnm} \sigma^l \tau^m$$

$$= \boxed{-[S^{j,A}(u), \tfrac{1}{2}\tau^n] = [S^{j,A}(u), S_j^n]}$$

$$T(u) = S^{1A}(u) S^{2A}(u) \cdots S^{NA}(u)$$

$$[T(u), S^n_{tot}] = \sum_{j=1}^{N} S^{1A} \cdots [S^{jA}, S^n_{tot}] \cdots S^{NA}$$

$$= -\sum_j L_1 \cdots [L_j, \frac{\tau^n}{2}] \cdots L_N = -[T(u), \tfrac{1}{2}\tau^n]$$

in the A space: $T_{\alpha\beta}(u)$ ← explicitly write the matrix element.

$$\Rightarrow [T_{\alpha\beta}(u), S^n_{tot}] = -[T(u) \tfrac{1}{2}\tau^n] = \frac{1}{2} [\tau^n_{\alpha\beta'} T_{\beta'\beta} - T_{\alpha\beta'} \tau^n_{\beta'\beta}]$$

$$T_N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$\Rightarrow$

$$\Rightarrow \begin{pmatrix} [A, S^n], & [B, S^n] \\ [C, S^n], & [D, S^n] \end{pmatrix} = -\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ \frac{1}{2}\tau^n \right]$$

$$-\frac{1}{2}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right] = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad -\frac{1}{2}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right], = -\frac{1}{2}\begin{pmatrix} B-C, & A-D \\ D-A, & C-B \end{pmatrix}$$

$$-\frac{1}{2}\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} & -i \\ i & \end{pmatrix}\right] = -\frac{1}{2}\begin{bmatrix} (B+C)i, & -(A-D)i \\ (D-A)i, & -(B+C)i \end{bmatrix}$$

$$\Rightarrow [A+D, S^n] = 0, \quad \text{for } n = 1, 2, 3$$

$$[B(u)\ S^3] = B(u), \quad [B(u), S^+] = [B(u), S^1] + i[B(u), S^2]$$

$$= -\frac{1}{2}(A-D) - \frac{1}{2}(A-D) = -(A-D)$$

or rewrite $\boxed{[S^n, A+D] = 0, \ [S^3, B(u)] = -B(u), \ [S^+, B(u)] = A(u) - D(u)}$

Obviously $S^+|0\rangle = 0, \quad S^3|0\rangle = \frac{N}{2}|0\rangle$

$$S^3 B(u) = B(u)(S^3 - 1) \Rightarrow$$

$$S^3 B(v_1)\cdots B(v_m)\,|0\rangle = B(v_1)(S^3-1)B(v_2)\cdots\ |0\rangle$$

$$= B(v_1)B(v_2)(S^3-2)\cdots\ |0\rangle = B(v_1)\cdots B(v_m)(S^3 - M)|0\rangle$$

$$= \frac{N-2M}{2}\ B(v_1)\cdots B(v_m)|0\rangle$$

$$[S^+, B(v_1)\cdots B(v_m)] = \sum_\alpha B(v_1)\cdots[S^+, B(v_\alpha)]\cdots B(v_m)$$

$$= \sum_\alpha B(v_1)\cdots(A(v_\alpha) - D(v_\alpha))\cdots B(v_m)$$

$$\Rightarrow \quad S^+ B(v_1)\cdots B(v_m)|0\rangle = [S^+, B(v_1)\cdots B(v_m)]|0\rangle = \sum_\alpha B(v_1)\cdots(A(v_\alpha) - D(v_\alpha))\cdots B(v_m)|0\rangle$$

by using $A(u)B(v) = \frac{1}{b(v-u)} B(v)A(u) - \frac{c(u-u)}{b(v-u)} B(u)A(v)$

$$D(u)B(v) = \frac{1}{b(u-v)} B(v)D(u) - \frac{c(u-v)}{b(u-v)} B(u)D(v)$$

$$D(u)|0\rangle = \prod_{j=1}^{N} b(u-u_j)|0\rangle, \quad A|0\rangle = |0\rangle.$$

We can expect that, the finial expression can be expressed as

$$S^+ B(v_1)\cdots B(v_m)|0\rangle = \sum_\alpha M_\alpha B(v_1)\cdots B(v_{\alpha-1}) B(v_{\alpha+1})\cdots B(v_m)|0\rangle$$

The coefficient of M_1 can be obtained as ← the term missing $\boxed{B(v_1)}$

$$(A(v_1) - D(v_1))\, B(v_2)\cdots B(v_m)|0\rangle$$

$$\Rightarrow M_1 = \prod_{\alpha=2}^{M} \frac{1}{b(v_\alpha - v_1)} - \frac{\prod_{j=1}^{N} b(v_1 - u_j)}{\prod_{\alpha=2}^{M} b(v_1 - v_2)} = 0. \leftarrow \text{Bethe antsatz Eq.}$$

Because all B commutate, we can do arbitary permutation of B. Thus from $M_1 = 0$, we obtain $M_\alpha = 0$

$$\Rightarrow \boxed{S^+ B(v_1)\cdots B(v_m)|0\rangle = 0}$$

Ground state solution:

Suppose $N =$ even, $m =$ odd, ~~set~~ First consider the limit of $c \to \infty$

$$k_j L = 2\pi I_j + 2\sum_{\beta=1}^{M} \tan^{-1} \frac{2\Lambda_\beta}{c}, \quad j = 1, \cdots N,$$

$\boxed{\begin{array}{l} I_j : \text{half integer} \\ J_\alpha : \text{integer} \end{array}}$

$$-2N \tan^{-1} \frac{2\Lambda_\alpha}{c} = 2\pi J_\alpha + 2\sum_{\beta=1}^{M} \tan^{-1} \frac{\Lambda_\beta - \Lambda_\alpha}{c}. \quad \alpha = 1, \cdots M.$$

$$\Rightarrow -2N \sum_{\alpha=1}^{M} \tan^{-1} \frac{2\Lambda_\alpha}{c} = 2\pi \sum_\alpha J_\alpha \Rightarrow 2\sum_{\alpha=1}^{M} \tan^{-1} \frac{2\Lambda_\alpha}{c} = \frac{2\pi}{N} \sum_\alpha J_\alpha$$

$$\Rightarrow k_j = \frac{2\pi}{L} (I_j + a), \quad a = \frac{1}{N} \sum_{\alpha=1}^{m} J_\alpha$$

$$E = \sum_{j=1}^{N} k_j^2 = \left(\frac{2\pi}{L}\right)^2 \sum_{j=1}^{N} (I_j + a)^2$$

where $a = 0$, I_j is closely packed around 0, $\rightarrow$ ground state.

at $c \neq \infty$, in the ground state I_j still takes the value as above.

J_α distribute symmetrically around origin, $\underline{\Lambda \text{ and } -\Lambda \text{ paired.}}$

$$2\pi I_j = L k_j + \sum_\alpha 2\tan^{-1} \frac{2k_j - 2\Lambda_\alpha}{c}$$

$$= L k_j + \sum_\alpha \tan^{-1} \frac{2k_j - 2\Lambda_\alpha}{c} + \tan^{-1} \frac{2k_j + 2\Lambda_\alpha}{c}$$

$$= L k_j + \sum_\alpha \tan^{-1} \frac{4k_j c}{c^2 + 4\Lambda_\alpha^2}.$$

whe Λ_α distribute evenly, we get energy minimum.

We consider $\overset{\text{ground state}}{N, L, M \to \infty}$, $\frac{N}{L}$, $\frac{M}{L}$ fixed, $\boxed{\begin{matrix} I_{j+1} - I_j = 1 \\ J_{\alpha+1} - J_\alpha = 1 \end{matrix}}$.

define $L\rho(k)\,dk$ = the number of k_j in the interval dk

$L\sigma(\Lambda)\,d\Lambda$ = the number of Λ_α in the interval $d\Lambda$

$$f = \frac{I}{L}, \quad g = \frac{J}{L}$$

$$\frac{df}{dk} = \rho(k), \quad \frac{dg}{d\Lambda} = \sigma(\Lambda)$$

where $\Theta(x) = -2\tan^{-1}\frac{x}{c}$

$$k_j L = 2\pi I_j + \sum_{\alpha=1}^{M} \Theta(2k_j - 2\Lambda),$$

$$\frac{d\Theta}{dx} = -2\,\frac{\frac{1}{c}}{1+\left(\frac{x}{c}\right)^2} = -2\,\frac{c}{c^2+x^2}$$

$$\Rightarrow \boxed{k = 2\pi f + \int_{-B}^{B} \Theta(2k-2\Lambda)\,\sigma(\Lambda)\,d\Lambda}$$

$$2\sum_{j=1}^{N} \tan^{-1}\frac{2(k_j - \Lambda_\alpha)}{c} = 2\pi J_\alpha + 2\sum_{\beta=1}^{M} \tan^{-1}\frac{\Lambda_\beta - \Lambda_\alpha}{c}$$

$$\Rightarrow \sum_{j=1}^{N} \Theta(2\Lambda_\alpha - 2k_j) = \underset{\substack{\uparrow \\ \text{up to a sign}}}{2\pi J_\alpha} + \sum_{\beta=1}^{M} \Theta(\Lambda_\alpha - \Lambda_\beta)$$

$$\boxed{\int_{-Q}^{Q} \Theta(2\Lambda - 2k)\,\rho(k)\,dk = -2\pi g + \int_{-B}^{B} \Theta(\Lambda - \Lambda')\,\sigma(\Lambda')\,d\Lambda'}$$

$$\frac{d}{dk} \quad \Rightarrow \quad 2\pi\rho(k) = 1 + \int_{-B}^{B} \frac{4c\,\sigma(\Lambda)\,d\Lambda}{c^2 + 4(k-\Lambda)^2}$$

$$\frac{d}{d\Lambda} \quad \Rightarrow \quad 2\pi\sigma(\Lambda) = -\int_{-B}^{B} \frac{2c\,\sigma(\Lambda')\,d\Lambda'}{c^2 + (\Lambda-\Lambda')^2} + \int_{-Q}^{Q} \frac{4c\,\rho(k)\,dk}{c^2 + 4(k-\Lambda)^2}$$

also constraint $\frac{N}{L} = \int_{-Q}^{Q} \rho(k)\, dk$, $\frac{M}{L} = \int_{-B}^{B} \sigma(\Lambda)\, d\Lambda$

we can solve $\rho, \sigma, Q, B \Rightarrow \frac{E}{L} = \int_{-Q}^{Q} k^2 \rho(k)\, dk$