OPE (SU(2))

Right mover:

$$\psi_R(-ix) = \frac{1}{\sqrt{L}} \sum_p e^{ipx} \psi_R(p) \quad , \quad p = \frac{2\pi}{L}(n+1/2),\ n \text{ is an integer}$$

$$\psi_R^\dagger(-ix) = \frac{1}{\sqrt{L}} \sum_p e^{-ipx} \psi_R^\dagger(p) \quad . \quad \text{(anti-pperiodical boundary condition)}$$

$$\psi_R(p) = \frac{1}{\sqrt{L}} \int dx\, e^{-ipx} \psi_R(x) \ , \qquad \psi_R^\dagger(p) = \frac{1}{\sqrt{L}} \int dx\, e^{ipx} \psi_R^\dagger(x)$$

$$\{\psi_R^\dagger(-ix), \psi_R(-iy)\} = \delta(x-y), \qquad \{\psi_R^\dagger(p), \psi_R(p')\} = \delta_{p,p'}$$

Left mover:

$$\psi_L(ix) = \frac{1}{\sqrt{L}} \sum_p e^{ipx} \psi_L(p) \quad , \quad p = \frac{2\pi}{L}(n+1/2)$$

$$\psi_L^\dagger(ix) = \frac{1}{\sqrt{L}} \sum_p e^{-ipx} \psi_L^\dagger(p)$$

$$\psi_L(p) = \frac{1}{\sqrt{L}} \int dx\, e^{-ipx} \psi_L(x), \qquad \psi_L^\dagger(p) = \frac{1}{\sqrt{L}} \int dx\, e^{ipx} \psi_L^\dagger(x)$$

$$\{\psi_L^\dagger(ix), \psi_L(iy)\} = \delta(x-y), \qquad \{\psi_L^\dagger(p), \psi_L(p')\} = \delta_{p,p'}$$

Current

$$J_R(q) = \int_0^L dx\, e^{-iqx} J_R(-ix) = \frac{1}{L} \int_0^L dx\, e^{-iqx} \sum_{pp'} e^{-i(p-p')x} \psi_R^\dagger(p)\psi_R(p')$$

$$= :\sum_p \psi_R^\dagger(p)\psi_R(p+q): \quad \text{(normalization)}$$

$$J_R(-ix) = \frac{1}{L} \sum_q J_R(q)\, e^{iqx}$$

$$J_L(q) = \int_0^L dx\, e^{-iqx} J_L(ix) = :\sum_p \psi_L^\dagger(p)\, \psi_L(p+q):$$

$$J_L(ix) = \frac{1}{L} \sum_q J_L(q)\, e^{iqx}$$

put in time

$$\psi_R(v\tau - ix) = \frac{1}{\sqrt{L}} \sum_p e^{-p(v_F\tau - ix)} \psi_R(p)$$

$$\psi_R^+(v\tau - ix) = \frac{1}{\sqrt{L}} \sum_p e^{p(v_F\tau - ix)} \psi_R^+(p)$$

$$\psi_L(v\tau + ix) = \frac{1}{\sqrt{L}} \sum_p e^{p(v_F\tau + ix)} \psi_L(p)$$

$$\psi_L^+(v\tau + ix) = \frac{1}{\sqrt{L}} \sum_p e^{-p(v_F\tau + ix)} \psi_L^+(p)$$

$$z = v\tau - ix, \qquad \bar{z} = v\tau + ix$$

$$i\partial_x = \partial_z - \partial_{\bar{z}}$$

$$\partial_\tau = \partial_z + \partial_{\bar{z}}$$

OPE in real space

J_L and J_R

$$:\psi_R^+(z)\psi_R(z): \equiv \lim_{\delta\to 0} \psi_R^+(z-i\delta)\psi_R(z) - \langle \psi_R^+(z-i\delta)\psi_R(z)\rangle$$

In Fourier transformation :

$$\psi_R^+(z-i\delta)\psi_R(z) = \frac{1}{L}\sum_q e^{-qz}\sum_p e^{-ip\delta}\psi_R^+(p)\psi_R(p+q)$$

$$= \frac{1}{L}\sum_{q\neq 0} e^{-qz} :\sum_p \psi_R^+(p)\psi_R(p+q): \;+ \frac{1}{L}\sum_p e^{-ip\delta} :\psi_R^+(p)\psi_R(p): \;+ \frac{1}{L}\sum_{p<0} e^{-ip\delta}$$

$$\frac{1}{L}\frac{1}{1-e^{i\frac{2\pi}{L}\delta}} = \frac{1}{-2\pi i\delta}$$

$$\underline{\langle \psi_R^+(z-i\delta)\psi_R(z)\rangle = \frac{1}{-2\pi i\delta}}$$

$$:\psi_L^+(\bar z)\psi_L(\bar z): \equiv \lim_{\delta\to 0} \psi_L^+(\bar z+i\delta)\psi_L(\bar z) - \langle \psi_L^+(\bar z+i\delta)\psi_L(\bar z)\rangle$$

$$\psi_L^+(\bar z+i\delta)\psi_L(\bar z) = \frac{1}{L}\sum_q e^{q\bar z}\sum_p e^{-ip\delta}\psi_L^+(p)\psi_L(p+q)$$

$$= \frac{1}{L}\sum_{q\neq 0} e^{q\bar z} :\sum_p \psi_L^+(p)\psi_L(p+q): \;+ \frac{1}{L}\sum_p :\psi_L^+(p)\psi_L(p): \;+ \frac{1}{L}\sum_{p>0} e^{-ip\delta} \hookrightarrow \frac{1}{2\pi i\delta}$$

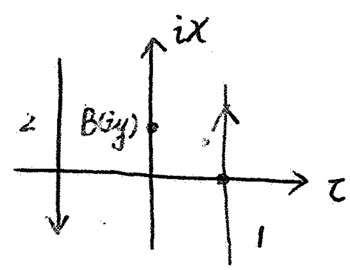
$$\underline{\langle \psi_L^+(\bar z+i\delta)\psi_L(\bar z)\rangle = \frac{1}{2\pi i\delta}}$$

② OPE

if $$A_R(z_1)B_R(z_2) = \frac{C_R(z_2)}{(z_1-z_2)^2} + \frac{D_R(z_2)}{(z_1-z_2)} + O(1) \;;$$

we are concen~~trated~~ in the equal time commutation,

$$[A_R(p), A_R(q)] = \underbrace{\quad}_{\lim_{\tau\to 0}} \int_0^l dx \int_0^l dy \; e^{-ipx} e^{-iqy} \underbrace{\{A(\tau-ix)B(-iy) - B(-iy)A(\underset{-\tau}{\widehat{-ix}})\}}_{\text{right time order}}$$



$$\int_0^L dx \quad e^{-ipx} \quad A(\tau-ix)B(-iy) - B(-iy)A(-\tau-ix)$$

$$= \frac{1}{i}\left[\int_1 dz \quad e^{pz} \quad A(z)B(-iy) + \int_2 dz \quad e^{pz} B(-iy)A(z)\right]$$

$$= \frac{1}{i}\oint dz \quad e^{pz} T_\tau[A(z)B(-iy)]$$

if we inte-prete $\quad A_R(z_1)B_R(z_2) = \frac{C_R(z_2)}{(z_1-z_2)^2} + \frac{D_R(z_2)}{z_1-z_2} + O(1)$

as time-ordered. to z_1, z_2

This is reasonable. under this definition. $A_R(z_1)A_R(z_2)$ is analytical (z_1-z_2).

For example.

$$-G_R(\tau, x) = \langle T_\tau \psi_R(\tau-ix)\psi_R^\dagger(0)\rangle$$

$$= \theta(\tau)\cdot\frac{1}{L}\sum_{p>0} e^{-p(\tau-ix)}\langle C_p C_p^\dagger\rangle - \theta(-\tau)\frac{1}{L}\sum_{p<0} e^{-p(\tau-ix)}\langle C_p^\dagger C_p\rangle$$

$$= \theta(\tau)\frac{1}{L}\cdot\frac{1}{1-e^{-\frac{2\pi}{L}(\tau-ix)}} - \theta(-\tau)\frac{1}{L}\frac{1}{1-e^{+\frac{2\pi}{L}(\tau-ix)}} \qquad (T=0k)$$

$$= \theta(\tau)\frac{1}{L}\cdot\frac{1}{+\frac{2\pi}{L}(\tau-ix)} - \theta(-\tau)\frac{1}{L}\frac{1}{-\frac{2\pi}{L}(\tau-ix)}$$

$$= \frac{1}{2\pi(\tau-ix)} = \frac{1}{2\pi z}$$ analyticaly only in the sense of time order (imaginary).

$$\left.\begin{aligned} G_R(z) &= \frac{-1}{2\pi}\cdot\frac{1}{L/\pi}\frac{1}{\sinh[\frac{\pi}{L}z]} \\ G_L(\bar z) &= \frac{-1}{2\pi}\frac{1}{L/\pi}\frac{1}{\sinh(\frac{\pi}{L}\bar z)} \end{aligned}\right\} \quad T>0K$$

Need check

$$[A_R(p), A_R(q)] = \int_0^L dy \oint \frac{dz}{i} e^{-iqy} e^{pz} \left[\frac{C_R(-iy)}{(z+iy)^2} + \frac{D_R(-iy)}{(z+iy)} + O(1) \right]$$

$$= \int_0^L dy \quad 2\pi \, e^{-iqy} \{[p e^{pz} C_R(-iy)]\}|_{z=-iy} + \{e^{pz} D_R(-iy)\}|_{z=-iy}$$

$$= 2\pi \int_0^L dy \quad p \, e^{-i(q+p)y} C_R(-iy) + e^{-i(q+p)y} D_R(-iy)$$

$$= 2\pi [p \, C(p+q) + D(p+q)] \qquad U(1) \text{ Kac-Moody algebra}$$

$$J_R(z_1) J_R(z_2) = :\psi_R^+ \psi_R:(z_1) :\psi_R^+ \psi_R(z_2): = -$$

$$= \frac{1}{(2\pi)^2} \frac{1}{(z_{12})^2} \qquad \text{b. } - \langle \psi_R(z_1)\psi^+(z_2)\rangle\langle\psi(z_2)\psi^+(z_1)\rangle \propto + \frac{1}{z_{12}} \cdot \frac{1}{+z_{12}} \propto \frac{1}{(2\pi)^2} \frac{1}{(z_{12})^2}$$

(two contraction)

$$+ [:\psi_R^+(z_1)\psi_R(z_2): + :\psi_R(z_1)\psi_R^+(z_2):] \frac{1}{+z_{12}} \frac{1}{2\pi}$$

$$+ :\psi_R^+(z_2)\psi_R(z_2)\psi_R^+(z_2)\psi_R(z_2): \quad \text{(All the sigularity has been moved to above)}$$

Although actually $:\psi_R^+(z_1)\psi_R(z_1)\psi_R^+(z_2)\psi_R(z_2): = \sum_{P_1 \to P_4} \frac{1}{L^2} e^{-i(P_1-P_2)z_1} e^{-i(P_2-P_3)z_2} C_R^+(P_1) C_R(P_2) C_R(P_3) C_R(P_4)$

$\sim O(z_{12})$ Try all the permutation, then the coeffient $\propto O(z_{12})$

$$:\psi_R^+(z_1)\psi_R(z_2): = :\psi_R^+(z_2)\psi_R(z_2): + z_{12} \, \partial_z\psi_R^+(z_2)\psi_R(z_2)$$

$$:\psi_R(z_1)\psi_R^+(z_2): = :\psi_R(z_2)\psi_R^+(z_2): + z_{12} \, \partial\psi_R(z_2)\psi_R^+(z_2)$$

$$J_R(z_1^*) J_R(z_2^*) = \frac{1}{(2\pi)^2} \frac{1}{z_{12}^2} + \frac{1}{2\pi}[(\partial_z \psi_R^+)\psi_R(z_2^*) - \psi_R^+(\partial_z\psi_R)(z_2)] + O(z_{12})$$

$$[J_R(p), J_R(q)] = \frac{1}{2\pi}$$

$$p \boxed{\int_0^L dy \;\; p \, e^{-i(q+p)y}} = Lp \, \delta_{p+q} \Rightarrow [J_R(p), J_R(q)] = \frac{Lp}{2\pi}\delta_{p+q}$$

$$[J_R(-ix), J_R(-iy)] = \frac{1}{L^2} \sum_{p,q} e^{+ipx+iqy} [J_R(p), J_R(q)] = \frac{1}{L}\frac{1}{2\pi} \sum_p p \, e^{+ip(x-y)} = (-i)\frac{\partial}{\partial x} \frac{1}{2\pi L} \sum_p e^{ip(x-y)}$$

$$= -i \frac{\partial}{\partial x} \frac{1}{2\pi} \delta(x-y)$$

For left moving

$$\underline{[A_L(p)\,,\,A_L(q)]} = \lim_{\tau\to 0}\int_0^L dx\int_0^L dy\ e^{-ipx}\,e^{-iqy}\,\left(A(\tau+ix)B(iy) - B(iy)A(-\tau+ix)\right)$$

$$= \lim_{\tau\to 0}\int_0^L \underbrace{dy}_{e^{-iqy}}\ \frac{1}{i}\oint d\bar z\ e^{-pz}\ T_\tau[A(\bar z)B(iy)] \longrightarrow \frac{C_L(\bar z_2)}{(\bar z_1-\bar z_2)^2} + \frac{D_L(\bar z_2)}{\bar z_1 - \bar z_2} + O(1)$$

$$= \int_0^L dy\ e^{-iqy}\ 2\pi\,[-p\,e^{-pz}\,C_L(iy) + \underbrace{D_R(iy)}_{e^{-pz}}]\Big|_{z=iy}$$

$$= \int_0^L dy\ e^{-i(q+p)y}\cdot 2\pi[-p\,C_L(iy) + D_L(iy)] = \underline{2\pi\,[-p\,C_L(p+q) + D_L(p+q)]}$$

$$J_L(\bar z_1)J_L(\bar z_2) = :\psi_L^\dagger(\bar z_1)\psi_L(\bar z_1)::\psi_L^\dagger(\bar z_2)\psi_L(\bar z_2): = \frac{1}{(2\pi)^2}\frac{1}{(\bar z_{12})^2} + \underline{:\psi_L^\dagger(\bar z_1)\psi_L(\bar z_2) + \psi_L(\bar z_1)\psi_L^\dagger(\bar z_2):}\,\frac{1}{2\pi\bar z_{12}}$$

$$+ :\psi_L^\dagger(\bar z_1)\psi_L(\bar z_1)\psi_L^\dagger(\bar z_2)\psi_L(\bar z_2):$$

$$:\psi_L^\dagger(\bar z_1)\psi_L(\bar z_2) + \psi_L(\bar z_1)\psi_L^\dagger(\bar z_2): = \bar z_{12}\,(\partial_{\bar z}\psi_L^\dagger(\bar z)\psi_L(\bar z) + \partial\psi_L\,\psi_L^\dagger(\bar z)]$$

$$\therefore\ J_L(\bar z_1)J_L(\bar z_2) = \frac{1}{(2\pi)^2}\frac{1}{(\bar z_{12})^2} + \frac{1}{2\pi}\,[\,\partial_{\bar z}\psi_L^\dagger\,\psi_L(\bar z) - \psi_L^\dagger\partial\psi_L(\bar z)] + O(\bar z_{12})$$

$$\underline{[J_L(p), J_L(q)] = 2\pi(-p)\int_0^L dy\ e^{-i(q+p)y} = \frac{2\pi L}{(2\pi)^2}(-p)\,\delta(p+q) = \frac{-pL}{2\pi}\,\delta(p+q)}$$

$$\underline{[J_L(+ix),\ J_L(iy)]} = \frac{1}{L^2}\int d\sum_{pq}[J_L(p), J_L(q)]\ e^{+ipx}\,e^{+iqy}\ \left(\frac{-pL}{2\pi}\right)\delta(p+q)$$

$$= \frac{1}{2\pi L}\sum_p e^{+ip(x-y)}(-p) = i\frac{\partial}{\partial x}\frac{1}{2\pi L}\sum_p e^{ip(x-y)} = \underline{\frac{i}{2\pi}\,\partial_x\delta(x-y)}$$

$$+\mathcal{L} = \psi_L^\dagger \frac{\partial}{\partial\tau}\psi_L + \psi_R^\dagger \frac{\partial}{\partial\tau}\psi_R + (-i\psi_R^\dagger \partial_x \psi_R + i\psi_L^\dagger \partial_x \psi_L) v_F$$

$$= v_F\left[\psi_L^\dagger\left(\frac{\partial}{\partial\tau} + i\partial_x\right)\psi_L + \psi_R^\dagger\left(\frac{\partial}{\partial\tau} - i\partial_x\right)\psi_R\right]$$

$$= 2v_F[\psi_L^\dagger \partial_z \psi_L + \psi_R^\dagger \partial_{\bar z}\psi_R]$$

$$\delta\mathcal{L} = 2v_F[\delta\psi_L^\dagger \partial_z\psi_L + \psi_L \partial_z \delta\psi_L + \delta\psi_R^\dagger \partial_{\bar z}\psi_R + \psi_R^\dagger \partial_{\bar z}\delta\psi_{R^+}]$$

$$= 2v_F(\partial_z\psi_L \delta\psi_L^\dagger - \sigma\partial_z\psi_L^\dagger \delta\psi_L) + (L\to R) = \cancel{4v_F(\delta\psi_L^\dagger \partial_z\psi_L + (L\to R))} \Rightarrow$$

motion equation $\quad \partial\psi_L/\partial z = \dfrac{\partial\psi_L}{\partial z} = 0, \qquad \partial\psi_R/\partial\bar z = \dfrac{\partial\psi_R^\dagger}{\partial\bar z} = 0$

$U(1)$ current: $\quad J_c(\tau,x) = \dfrac{\delta\mathcal{L}}{\delta(\partial_c\psi_\sigma)}\psi_\sigma \qquad \psi_L \to e^{-i\alpha}\psi_L \qquad \psi_R \to e^{-i\alpha}\psi_R$

$$\mathcal{L} \to e^{i\alpha}\psi_L^\dagger[e^{-i\alpha}\partial_\tau]\psi_L + (-i\partial_\tau\alpha\, \psi_L^\dagger\psi_L) + \cdots + (-i\partial_\tau\alpha\, \psi_R^\dagger\psi_R)$$
$$+ (-\psi_R^\dagger\psi_R\partial_x\alpha) + (\psi_L^\dagger\psi_L)\partial_x\alpha$$

$$\delta\mathcal{L} = -i\partial_\tau\alpha(\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R) + \partial_x\alpha[-\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L]$$

$$= \partial[\alpha\cdots] + \{i\partial_\tau[\psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R] + \partial_x[\psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L]\}\cdot\alpha$$

$$\Rightarrow \quad J_\tau = \psi_L^\dagger\psi_L + \psi_R^\dagger\psi_R \qquad \boxed{\partial_\tau J_\tau + i\partial_x J_x = 0}$$
$$J_x = -i[\psi_R^\dagger\psi_R - \psi_L^\dagger\psi_L]$$

$$\partial_\tau J_\tau + \partial_x J_x = \partial_\tau\, \frac{1}{2}(\partial_\tau + i\partial_x)(\underbrace{J_\tau - iJ_x}_{J_L = L^\dagger L}) + \frac{1}{2}(\partial_\tau - i\partial_x)(\underbrace{J_\tau + iJ_x}_{J_R = R^\dagger R})$$

$$= \underbrace{(\partial_{\bar z} J_L + \partial_{\bar z} J_R)}_{2} = 0$$

$$H_0 = v_F \int_0^L dx \quad :\psi_L^\dagger i\partial_x \psi_L: \; - :\psi_R^\dagger i\partial_x \psi_R: = v_F \int_0^L dx \left[:\psi_L^\dagger (\partial_z - \partial_{\bar z}) \psi_L: - :\psi_R^\dagger (\partial_z - \partial_{\bar z}) \psi_R^\dagger \right).$$

$$\hat{=}\, v_F \int_0^L dx \quad \psi_R^\dagger \partial_z \psi_R^\dagger + \psi_L^\dagger \partial_{\bar z} \psi_L$$

$$J_L(\bar z_1) J_L(\bar z_2) = \frac{1}{(2\pi)^2} \frac{1}{\bar z_{12}^2} + \frac{1}{2\pi} \left[\partial_{\bar z} \psi_L^\dagger \psi_L + \psi_L^\dagger \partial_{\bar z} \psi_L \right]$$

$$J_R(z_1) J_R(z_2) = \frac{1}{(2\pi)^2} \frac{1}{z_{12}} + \frac{1}{2\pi} \left[\partial_z \psi_R^\dagger \psi_R - \psi_R^\dagger \partial_z \psi_R \right]$$

$$\therefore \quad H_0 = -v_F \int_0^L dx \; (-\pi) (J_R J_R + J_L J_L) = v_F \pi \int_0^L dx :J_R J_R + J_L J_L:$$

SU(2) Kac-Mody algebra:

$$J_R(z_1)J_R(z_2) = :\psi^\dagger_{R\alpha}(z_1)\psi_{R\alpha}(z_1)::\psi^\dagger_{R\beta}(z_2)\psi_{R\beta}(z_2): = \frac{2}{(z_{12})^2(2\pi)^2}$$

$$+ :\psi_{R\alpha}(z_1)\psi^\dagger_{R\alpha}(z_2) + \psi^\dagger_{R\alpha}(z_1)\psi_{R\beta}(z_2): \frac{1}{z_{12}} + :\psi^\dagger_{R\alpha}(z_1)\psi_{R\alpha}(z_1)\psi^\dagger_{R\beta}(z_2)\psi_{R\beta}(z_2):$$

$$= \frac{2}{(2\pi)^2 z_{12}^2} + \frac{1}{2\pi}\left[\partial_z\psi^\dagger_{R\alpha}(z_2)\psi_{R\alpha}(z_2) - \psi^\dagger_{R,\alpha}(z_2)\partial_z\psi_{R\alpha}(z_2)\right] + :\psi^\dagger_{R\alpha}(z_1)\psi_{R\alpha}(z_1)\psi^\dagger_{R\beta}(z_2)\psi_{R\beta}(z_2):$$

<u>(U(1) current)</u>

$$J^a_R(z_1)J^b_R(z_2) = \frac{1}{4}:\psi^{\dagger\alpha}_R(z_1)\,\sigma^a_{\alpha\beta}\,\psi_{R\beta}(z_1)::\psi^\dagger_{R\gamma}(z_2)\,\sigma^b_{\gamma\delta}\,\psi_{R\delta}(z_2):$$

$$= \frac{1}{4}:\sigma^a_{\alpha\beta}\sigma^b_{\beta\alpha}:\frac{1}{z_{12}^2(2\pi)} + \frac{1}{4}:\psi^\dagger_{R\alpha}(z_1)\,\sigma^a_{\alpha\beta}\,\sigma^b_{\beta\sigma}\,\psi_{R\delta}(z_2) - \psi^\dagger_{R\gamma}(z_2)\,\sigma^b_{\gamma\delta}\,\sigma^a_{\delta\beta}\,\psi_{R\beta}(z_1):\frac{1}{2\pi z_{12}}$$

$$+\frac{1}{4}:\psi^\dagger_{R,\alpha}(z_1)\,\sigma^a_{\alpha\beta}\psi_{R,\beta}(z_1)\,\psi^\dagger_{R\gamma}(z_2)\,\sigma^b_{\gamma\delta}\,\psi_{R\delta}(z_2):$$

$$= \frac{1}{4}\frac{tr(\sigma^a\sigma^b)}{(2\pi z_{12})^2} + \frac{1}{4}:\partial_z\psi^\dagger_{R\alpha}(z_2)\psi_{R\delta}(z_2)\;\sigma^a_{\alpha\beta}\sigma^b_{\beta\sigma} - \underbrace{\psi^\dagger_{R\gamma}(z_2)\,\partial_z\psi_{R\beta}(z_2)\;\sigma^b_{\gamma\delta}\sigma^a_{\delta\beta}}_{\cancel{2i\epsilon^{abc}\sigma^c}}:\frac{1}{2\pi}$$

$$+\frac{1}{4}:\psi^\dagger_{R\alpha}(z_1)\psi_{R,\beta}(z_1)\psi^\dagger_{R\gamma}(z_2)\psi_{R\delta}(z_2):\;\sigma^a_{\alpha\beta}\,\sigma^b_{\gamma\delta}$$

$$\sigma^a\sigma^b = i\epsilon^{abc}\sigma^c + \delta^{ab}$$

$$\cancel{= \frac{1}{2}\frac{\delta^{ab}}{(2\pi z_{12})^2} + \frac{1}{2}\frac{1}{(2\pi z_{12})}\,i\epsilon^{abc}:\partial_z\psi^\dagger_{R\alpha}(z_2)\psi_{R\beta}(z_2)\,\sigma^c_{\alpha\beta}:}$$

$$+\frac{1}{4}\,2:\psi^\dagger_{R\alpha}\psi_{R\beta}(z_2):\,i\sigma^c_{\alpha\beta}\cdot\epsilon^{abc}\;\frac{1}{2\pi z_{12}}$$

$$= \frac{(1/2)\delta^{ab}}{(2\pi z_{12})^2} + \frac{(1/2)}{2\pi z_{12}}\,i\epsilon^{abc}:\psi^\dagger_{R\alpha}\sigma^c_{\alpha\beta}\psi_{R\beta}: + \frac{1}{4}:\left[\partial_z\psi^\dagger_{R\alpha}(\sigma^a\sigma^b)_{\alpha\beta}\psi_{R\beta}(z_2) - \psi^\dagger_{R\alpha}(\sigma^b\sigma^a)_{\alpha\beta}\,\partial_z\psi_{R\beta}(z_2)\right]\frac{1}{2\pi}$$

$$+\frac{1}{4}\cdot\psi^\dagger_{R\alpha}(z_1)\psi_{R\beta}(z_1)\psi^\dagger_{R\gamma}(z_2)\psi_{R\delta}(z_2):\;\sigma^a_{\alpha\beta}\sigma^b_{\gamma\delta}$$

$$= \frac{(1/2)\delta^{ab}}{(2\pi z_{12})^2} + \frac{1}{2\pi z_{12}}\,i\epsilon^{abc}J^c_R(z_2) + \frac{1}{4}\delta^{ab}\left(:\partial_z\psi^\dagger_{R\alpha}\psi_{R\alpha}(z_2) - \psi^\dagger_{R\alpha}\partial_z\psi_{R\beta}(z_2):\right)\underbrace{}_{(1/2\pi)} + \frac{i\epsilon^{abc}}{4}\underbrace{}_{(1/2\pi)}\left[\partial_z\psi^\dagger_{R\alpha}\sigma^c_{\alpha\beta}\psi_{R\beta} + \psi^\dagger_{R\alpha}\sigma^c\partial_z\psi_{R\beta}\right]$$

$$+\cdots$$

$$J_R^a(z_1)\, J_R^b(z_2) = \frac{(1/2)\delta^{ab}}{4\pi^2 z_{12}^2} + \frac{1}{2\pi z_{12}}\, i\varepsilon^{abc} J_R^c(z_2)$$

$$+ \frac{1}{4}\cdot\frac{\delta^{ab}}{2\pi}\left(:\partial_z \psi^\dagger_{R,a}\psi_{Ra}:(z_2) - :\psi^\dagger_{R,a}\partial_z\psi_{Ra}:(z_2)\right) + i\varepsilon^{abc}\cdot\frac{1}{2\pi}\,\frac{1}{2}\,\partial_z J_R^c(z_2)$$

$$+ \frac{1}{4} : \psi^{\dagger\alpha}_R(z_2)\psi_{R\beta}(z_2)\psi^{\dagger\gamma}_R(z_2)\psi_{R\delta}(z_2) : (\sigma^a)_{\alpha\beta}(\sigma^b)_{\gamma\delta}$$

Similarly

$$J_L^a(\bar z_1)\, J_L^b(\bar z_2) = :\psi^\dagger_{L,\alpha}(\bar z_1)\left(\frac{\sigma^a}{2}\right)_{\alpha\beta}\psi_{L\beta}(\bar z_1) :: \psi^\dagger_{L\gamma}(\bar z_2)\left(\frac{\sigma^b}{2}\right)_{\gamma\delta}\psi_{L\delta}(\bar z_2):$$

$$= \frac{1}{4}\frac{\mathrm{tr}\,\sigma^a\sigma^b}{(2\pi\bar z_{12})^2} + \frac{1}{4}\frac{1}{\bar z_{12}(2\pi)} : \psi^\dagger_{L\alpha}(\bar z_1)\psi_{L\delta}(\bar z_2)(\sigma^a\sigma^b)_{\alpha\delta} - \psi^\dagger_{L\gamma}(\bar z_2)\psi_{L\beta}(\bar z_1)(\sigma^b\sigma^a)_{\gamma\beta}:$$

$$+ :\psi^\dagger_{L\alpha}(\bar z_2)\psi_{L\beta}(\bar z_2)\psi^\dagger_{\gamma}(\bar z_2)\psi_\delta(\bar z_2): \frac{(\sigma^a)_{\alpha\beta}(\sigma^b)_{\gamma\delta}}{4}$$

$$= \frac{\frac{1}{2}\delta^{ab}}{(2\pi\bar z_{12})^2} + \frac{1}{2\pi\bar z_{12}} : \psi^\dagger_{L\alpha}\underbrace{\frac{\sigma^c_{\alpha\beta}}{2}}_{i\varepsilon^{abc}}\psi_{L\beta}:(\bar z_2) + \frac{1}{4\cdot 2\pi} :\partial_{\bar z}\psi^\dagger_{L\alpha}\psi_{L\beta}(\sigma^a\sigma^b)_{\alpha\beta}$$

$$- \psi^\dagger_{L\alpha}\partial_{\bar z}\psi_L(\sigma^b\sigma^a)_{\gamma\beta}:$$

$$+ \cdots\cdots$$

$$\boxed{\begin{aligned} &= \frac{\frac{1}{2}\delta^{ab}}{(2\pi\bar z_{12})^2} + \frac{i\varepsilon^{abc}}{2\pi\bar z_{12}} J_L^c(\bar z_2) + \frac{1}{2\pi}\cdot\frac{1}{4} : \partial_{\bar z}\psi^\dagger_{L\alpha}\psi_{L\alpha} - \psi^\dagger_{L\alpha}\partial_{\bar z}\psi_{L\alpha}:(\bar z_2) \\ &+ \frac{i\varepsilon^{abc}}{2\cdot 2\pi} : \partial_{\bar z} J_L^c(\bar z_2) + :\psi^\dagger_{L\alpha}\psi_{L\beta}\psi^\dagger_{L\gamma}\psi_{L\delta}: \frac{(\sigma^a)_{\alpha\beta}(\sigma^b)_{\gamma\delta}}{4}\end{aligned}}$$

$$:J_R J_R(z): = \frac{1}{2\pi}\left[(\partial_z \psi^+_{R,\alpha}\psi_{R,\alpha})(z_2) - \psi^+_{R,\alpha}\partial_z\psi_{R\alpha}(z_2)\right] + :\psi^+_{R\alpha}\psi_{R\alpha}\psi^+_{R\beta}\psi_{R\beta}:(z_2)$$

$$:\vec{J}_R \vec{J}_R:(z) = \frac{3}{4}\cdot\frac{1}{2\pi}\left(:(\partial_z\psi^+_{R\alpha}\psi_{R\alpha})(z_2) - (\psi^+_{R,\alpha}\partial_z\psi_{R\alpha})(z_2):\right]$$

$$+\frac{1}{4} :\psi^+_{R\alpha}\psi_{R\beta}\psi^+_{R\gamma}\psi_{R\delta}:(z_2) \qquad \underline{\sigma^a_{\alpha\beta}\sigma^a_{\gamma\delta} \to -\delta_{\alpha\beta}\delta_{\gamma\delta} + 2\delta_{\beta\gamma}\delta_{\alpha\delta}}$$

$$\cdot\frac{1}{4}: -\psi^+_{R\alpha}\psi_{R\alpha}\psi^+_{R\beta}\psi_{R\beta} + 2\psi^+_{R\alpha}\psi_{R\beta}\psi^+_{R\beta}\psi_{R\alpha}: \to -\frac{3}{4}:\psi^+_{R\alpha}\psi_{R\alpha}\psi^+_{R\beta}\psi_{R\beta}:$$

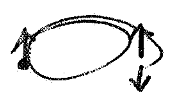
$$\therefore \quad \frac{1}{4}J_RJ_R + \frac{1}{3}\vec{J}_R\vec{J}_R = \frac{1}{2}\cdot\frac{1}{2\pi}\left[(\partial_z\psi^+_{R\alpha})\psi_{R\alpha}(z_2) - \psi^+_{R\alpha}\partial_z\psi_{R\alpha}(z_2)\right]$$

$$\underline{H_0} = -v_F\int_0^L dx \quad \psi^+_{R\sigma}\partial_z\psi_{R\sigma} + \psi^+_{L\sigma}\partial_z\psi_{L\sigma}$$

$$= v_F(2\pi)\int_0^L dx \quad \frac{1}{4}J_RJ_R + \frac{1}{3}\vec{J}_R\vec{J}_R + (R\leftrightarrow L)$$ Free hamiltonian.

$$= \int_0^L dx \quad \frac{\pi v_F}{2}\left[J_RJ_R + J_LJ_L\right] + \frac{2\pi v_F}{3}\left[\vec{J}_R\cdot\vec{J}_R + \vec{J}_L\cdot\vec{J}_L\right]$$

Actually, each term $\underline{J^z_R\cdot J^z_R,\ J^x_R\cdot J^x_R\ \cdot J^y_R\cdot J^y_R}$ are $SU(2)$ invariant, but not explicitly.



$$:J^z_R J^z_R: = :J^x_R J^x_R: = :J^y_R J^y_R:$$

For $\sigma^z_{\alpha\beta}\sigma^z_{\gamma\delta} = \delta_{\alpha\beta}\delta_{\gamma\delta}\,\epsilon_{\alpha\gamma}$:

$$J^R_z\cdot J^R_z \to -\frac{1}{4}:\psi^+_{R\uparrow}\psi_{R\uparrow}\psi^+_{R\downarrow}\psi_{R\downarrow}:\cdot(2) = -\frac{1}{2}:n_\uparrow n_\downarrow:$$

$$J^R_x\cdot J^R_x \to \frac{1}{4}:\psi^+_{R\uparrow}\psi_{R\downarrow}\psi^+_{R\downarrow}\psi_{R\uparrow}:(\times 2) = -\frac{1}{2}:n_\uparrow n_\downarrow:$$ other term such as

$$J^R_y\cdot J^R_y \to \frac{1}{4}: \qquad :(\times 2) = -\frac{1}{2}:n_\uparrow n_\downarrow: \qquad :\psi^+_{R\uparrow}\psi_{R\downarrow}\psi^+_{R\uparrow}\psi_{R\downarrow}: = 0$$

more physically. $\underset{z\ z+\epsilon}{\uparrow\uparrow}$ $\downarrow\downarrow$ $\updownarrow$ in all this states $\boxed{S_x^2 = S_y^2 = S_z^2}$, $\boxed{\text{but } \uparrow\downarrow \text{ has}}$ a $2k_F$ $J_L\cdot J_R$ componet.

$$[J_R^a(p), J_R^b(q)] = \cancel{-}\, 2\pi \int_0^L dy \; p\, e^{-i(q+p)y} \cdot \frac{1/2}{4\pi^2} + 2\pi \int_0^L dy \; e^{-i(q+p)y} \cdot \frac{i\varepsilon^{abc}}{2\pi} J_R(-iy)$$

$$= \frac{L}{4\pi} \cdot p\, \delta_{p+q,0}\, \delta_{ab} + i\varepsilon^{abc} J_R^c(q+p)$$

$$[J_L^a(p), J_L^b(q)] = 2\pi \int_0^L dy (-p) e^{-i(q+p)y}\; \tfrac{1}{2}/4\pi^2 + 2\pi \int_0^L dy\; e^{-i(q+p)y} \cdot \frac{i\varepsilon^{abc}}{2\pi} J_{\cancel{R}\,L}^c(-iy)$$

$$= \frac{L}{4\pi}(-p)\, \delta_{p+q,0}\, \delta_{ab} + i\varepsilon^{abc} J_L^c(q+p)$$

$$[J_R^a(-ix), J_R^b(-iy)] = \frac{1}{L^2} \sum_{pq} e^{ipx+iqy} [J_R(p), J_R(q)]$$

$$= \frac{1}{L^2} \sum_{p,q} p\cdot\delta_{p+q,0} \frac{L}{4\pi} \cdot e^{ipx+iqy} + \frac{1}{L^2} \sum_{p,q} e^{ipx+iqy}\; i\varepsilon^{abc} J_R^c(q+p)$$

$$= \frac{1}{4\pi L} \sum_p p \cdot e^{ip(x-y)} + \frac{1}{L^2} \sum e^{i[(p+q)\frac{(x+y)}{2} + \frac{(p-q)}{2}(x-y)]}\; i\varepsilon^{abc} J_R^c$$

$$= \frac{1}{\cancel{2}4\pi L} \sum_p (-i)\frac{\partial}{\partial x} e^{ip(x-y)} + \cancel{\oplus} \left(\frac{1}{L}\sum e^{i(p+q)\frac{x+y}{2}} \cdot J_R^c(p+q)\right) \frac{1}{L} \sum (p\, e^{\frac{i(p-q)}{2}\frac{(x+y)}{2}}]$$

$$\swarrow\; J_R^c\left(-i\frac{x+y}{2}\right) \quad \delta(x-y)$$

$$= \cancel{\ominus} \frac{-i}{4\pi} \partial_x \delta(x-y)\, \delta_{ab} + i\varepsilon^{abc} J_R^c(-ix)\delta(x-y)$$

$$[J_L^a(ix), J_L^b(iy)] = \frac{1}{L^2}\sum_{pq} e^{ipx+iqy} [J_L(p), J_L(q)] = \frac{1}{L^2}\sum_{pq} -p\,\delta_{p+q,0}\frac{L}{4\pi} e^{ipx+iqy}$$

$$+ \frac{1}{L^2}\sum_{pq} e^{ipx+iqy}\; i\varepsilon^{abc} J_L^c(p+q)$$

$$= \frac{i}{4\pi}\partial_x \delta(x-y) + i\varepsilon^{abc} J_L^c(ix)\delta(x-y)$$