Bosonization summary — elementary knowledge

* useful identities :

① $e^{-B} e^{A} e^{B} = A + [A,B] + \frac{1}{2!}[[A,B],B] + \cdots$

② if define $C = [A,B]$, and C commute with A, B, then $\Rightarrow$

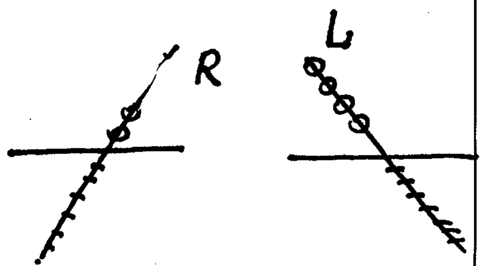
$$e^{-B} A e^{B} = A + C, \quad \text{or} \quad [A, e^{B}] = C e^{B}$$

$$e^{A} e^{B} = e^{A+B} e^{C/2} = e^{B} e^{A} \cdot e^{C}$$

• mode expansion

$$\psi_\nu(x) = \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{+\infty} C_{k,\nu}\, e^{ikx}, \qquad \nu = R, L$$

define for $q>0$, R- branch



$$\begin{cases} b_q^{+} = \dfrac{i}{\sqrt{n_q}} \displaystyle\sum_{k=-\infty}^{+\infty} C^{+}_{R,k+q}\, C_{R,k} \\ b_q = \dfrac{-i}{\sqrt{n_q}} \displaystyle\sum_{k=-\infty}^{+\infty} C^{+}_{R,k-q}\, C_{R,k} \end{cases}$$

for $q<0$, L - branch

$$n_q = \frac{|q|}{2\pi/L}$$

$$\begin{cases} b_q^{+} = \dfrac{-i}{\sqrt{n_q}} \displaystyle\sum_{k=-\infty}^{+\infty} C^{+}_{L,k+q}\, C_{L,k} \\ b_q = \dfrac{i}{\sqrt{n_q}} \displaystyle\sum_{k=-\infty}^{+\infty} C^{+}_{L,k-q}\, C_{L,k} \end{cases}$$

$\Rightarrow$ check $[b_q, b^{+}_{q'}] = \delta_{qq'}$

from momentum to coordinate space

$$\begin{cases} \varphi_R(x) = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} b_q \, e^{+iqx - aq/2} \\ \varphi_R^\dagger(x) = \sqrt{\frac{1}{4\pi}} \sum_{q>0} \frac{1}{\sqrt{n_q}} b_q^\dagger \, e^{-iqx - aq/2} \end{cases}$$

← Convergence factor
a - short distance cut off

$$\begin{cases} \varphi_L(x) = \sqrt{\frac{1}{4\pi}} \sum_{q<0} \frac{1}{\sqrt{n_q}} b_q \, e^{iqx + aq/2} \\ \varphi_L^\dagger(x) = \sqrt{\frac{1}{4\pi}} \sum_{q<0} \frac{1}{\sqrt{n_q}} b_q^\dagger \, e^{-iqx + aq/2} \end{cases}$$

chiral version

→ check commutation relation

$$[\varphi_R(x), \varphi_R^\dagger(x')] = \frac{-1}{4\pi} \ln \frac{2\pi}{L} (a - i(x-x'))$$

$$[\varphi_L(x), \varphi_L^\dagger(x')] = \frac{-1}{4\pi} \ln \frac{2\pi}{L} (a + i(x-x'))$$

chiral boson creation / annillation field

→ Chiral field

$$\begin{cases} \phi_R(x) = \varphi_R(x) + \varphi_R^\dagger(x) + \frac{\sqrt{\pi}\, x}{L} \hat{N}_R \, , & x \to x - v_f t \\ \phi_L(x) = \varphi_L(x) + \varphi_L^\dagger(x) + \frac{\sqrt{\pi}\, x}{L} \hat{N}_L \, , & x \to x + v_f t \end{cases}$$

The term proptotional to $\hat{N}_L, \hat{N}_R$ belongs to $q=0$. component

check ϕ_R, as $q \to 0$: $\sqrt{\frac{1}{4\pi}} \lim_{q\to 0} \frac{1}{\sqrt{n_q}} b_q$

Some subtlty need to be clarified !

$\boxed{\text{check commutator}}$

$$[\varphi_R(x), \varphi_R^\dagger(x')] = \frac{1}{4\pi} \sum_{q>0} \frac{1}{n_q} e^{iq(x-x') - aq} \quad , \quad q = \frac{2\pi}{L} n_q$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= \frac{1}{4\pi} \sum_{n_q=1}^{\infty} \frac{1}{n_q} \left[e^{i\frac{2\pi}{L}[(x-x') - a]}\right]^{n_q}$$

$$= -\frac{1}{4\pi} \ln\left[1 - e^{\left(i\frac{2\pi}{L}[(x-x') - a]\right)}\right) = -\frac{1}{4\pi} \ln\left[\frac{2\pi}{L}(a - i(x-x'))\right]$$

$$[\varphi_L(x), \varphi_L^\dagger(x')] = \frac{1}{4\pi} \sum_{q<0} \frac{1}{n_q} e^{iq(x-x') - aq}$$

$$= \frac{1}{4\pi} \sum_{n_q=1}^{\infty} \frac{1}{n_q} \left[e^{i -\frac{2\pi}{L}(x-x') - a}\right]^{n_q} = -\frac{1}{4\pi} \ln\left(1 - e^{i\left(-\frac{2\pi}{L}(x-x') - a\right)}\right]$$

$$= -\frac{1}{4\pi} \ln \frac{2\pi}{L}(a + i(x-x'))$$

check the limit $\varphi_R(x) + \varphi_R^\dagger(x)$ for the mode $q \to 0$

$$\lim_{q\to 0^+} \sqrt{\frac{1}{4\pi}} \frac{1}{\sqrt{n_q}} \frac{i}{\sqrt{n_q}} \frac{\hat{N}_R\left[-e^{iqx - aq/2} + e^{-iqx - aq/2}\right]}{2}$$

$q^{\infty} = 0$ only one mode

$$= \sqrt{\frac{1}{4\pi}} \frac{2\pi}{L} \frac{1}{|q|} \hat{N}_R \quad |q| x = \frac{\sqrt{\pi}\, x}{L} \hat{N}_R$$

average from left + right

also $\varphi_L(x) + \varphi_L^\dagger(x)$ for the mode $q \to 0$

$$\lim_{q\to 0^-} \sqrt{\frac{1}{4\pi}} \frac{1}{\sqrt{n_q}} \frac{b_q e^{iqx} + b_q^\dagger e^{-iqx}}{2} = \sqrt{\frac{1}{4\pi}} \frac{2\pi}{L} \frac{1}{|q|} \hat{N}_L \frac{+ie^{iqx} - ie^{-iqx}}{2}$$

$$= \sqrt{\frac{1}{4\pi}} \frac{2\pi}{L} \frac{1}{|q|} \hat{N}_L (-q) = \frac{\sqrt{\pi}}{L} x \hat{N}_L$$

$-q = |q|$

Senechal P_{13}

more comments on mode expansion:

- the zero mode, is ill defined, which has to be ~~treated~~ separately.

the zero mode spoils left-right

- more rigerously, we should ~~and~~ define $\phi = \phi_R + \phi_L$ as

$$\phi(x,t) = \varphi_R(x) + \varphi_R^\dagger(x) + \varphi_L(x) + \varphi_L^\dagger(x)$$

$$Q_0 + \frac{\Pi_0 v_f t}{L} + \frac{\tilde{\Pi}_0 x}{L}$$

$\phi(x,t)$ should be a compact field, let us set the radius R (to be determined) to be general here i.e. $\phi(x,t)$ and $\phi(x,t)+2\pi R$ is identical.

Q is the zero momentum ϕ. (zero mode)

i.e. all the point x, behaves the same way as a rigid body.

Π_0 is the momentum conjugate to it. Π_0 takes value $\frac{n 2\pi}{2\pi R} = \frac{n}{R}$.

- winding numbers are allowed precisely by $\tilde{\Pi}_0$.

$\varphi_R, \varphi_R^\dagger, \varphi_L, \varphi_L^\dagger$ are regular part, which gives rise to

$$\phi(x=L) = \phi(x=0).$$

but $\frac{\tilde{\Pi}_0 x}{L}$ not, as $\tilde{\Pi}_0 = 2m\pi R$, where m is total particle number. $\boxed{\tilde{R} = \frac{1}{2\pi R}}$

take value of

we can define a variable $\tilde{Q}_0$ conjugate to $\tilde{\Pi}_0$, which is compact radiaus $\frac{2\pi}{2\pi \tilde{R}} = 2\pi R$

$$\phi_R(x,t) = \frac{Q_0 - \tilde{Q}_0}{2} + \frac{\tilde{\Pi}_0 - \Pi_0}{2L}(x - v_f t) + \varphi_R(x) + \varphi_R^{\dagger}(x)$$

$$\phi_L(x,t) = \frac{Q_0 + \tilde{Q}_0}{2} + \frac{\tilde{\Pi}_0 + \Pi_0}{2}(x + v_f t) + \varphi_L(x) + \varphi_L^{\dagger}(x).$$

the zero mode L-R decomposition is not perfect

in the periodical boundary quantization. $[\phi_R, \phi_L] = 0$,

in order to ensure ψ_R and ψ_L anticommute, we have to introduce two Klein factors for left and right movers.

- If with infinite line, and vanishing boundary condition at $x = \pm\infty$

we will have $[\phi_R, \phi_L] = \frac{i}{4}$.

Senechal P21 foot note

Let us use it

~~check chiral field~~ : commutaters

$$\begin{cases} [\phi_R(x), \phi_R(x')] = \frac{i}{4}\,\mathrm{sgn}(x-x') \\ [\phi_L(x), \phi_L(x')] = -\frac{i}{4}\,\mathrm{sgn}(x-x') \\ [\phi_R(x), \phi_L(x')] = \frac{i}{4} \end{cases}$$

← important for anti-commutation for fermions ψ_R, ψ_L. no need for introducing Klein factors

↓

$$\begin{cases} \phi(x) = \phi_R(x) + \phi_L(x) \\ \theta(x) = \phi_R(x) - \phi_L(x) \\ [\phi(x), \phi(x')] = [\theta(x), \theta(x')] = 0 \\ [\phi(x), \theta(x')] = \end{cases}$$

阶跃函数

$$-i\,\theta(x'-x) = \begin{cases} 0 & x' < x \\ -i & x' > x \end{cases}$$

↓ bosonic observable

$$\rho_R(x) = \qquad = \sqrt{\frac{1}{\pi}}\,\partial_x \phi_R(x)$$

$$\rho_L(x) = \qquad = \sqrt{\frac{1}{\pi}}\,\partial_x \phi_L(x)$$

$$\rightarrow \quad \rho(x) = \rho_R(x) + \rho_L(x) = \frac{1}{\sqrt{\pi}}\,\partial_x \phi$$

$$j(x) = v_f\,(\rho_R(x) - \rho_L(x)) = \frac{v_F}{\sqrt{\pi}}\,\partial_x \theta$$

check formula

$$[\phi_R(x), \phi_R(x')] = [\varphi_R(x) + \varphi_R^\dagger(x), \varphi_R(x') + \varphi_R^\dagger(x')] = [\varphi_R(x), \varphi_R^\dagger(x')] + [\varphi_R^\dagger(x), \varphi_R(x')]$$

$$= \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a - i(x-x')) - \left(\frac{-1}{4\pi}\right) \ln \frac{2\pi}{L}(a + i(x-x'))$$

$$= \frac{-1}{4\pi} \ln \frac{a - i(x-x')}{a + i(x-x')} = \qquad = \frac{i}{4} \operatorname{sgn}(x-x')$$

if $(x-x') > 0 \Rightarrow \frac{-1}{4\pi}\left[-\frac{\pi}{2} - \frac{\pi}{2}\right] i = \frac{i}{4}$

$(x-x') < 0 \Rightarrow \frac{-1}{4\pi}\left[\frac{\pi}{2} \times 2\right] i = -\frac{i}{4}$

$$[\phi_L(x), \phi_L(x')] = [\varphi_L(x), \varphi_L^\dagger(x')] + [\varphi_L^\dagger(x), \varphi_L(x')]$$

$$= \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a + i(x-x')) - \left(\frac{-1}{4\pi}\right) \ln \frac{2\pi}{L}(a - i(x-x'))$$

$$= \frac{-1}{4\pi} \ln \frac{a + i(x-x')}{a - i(x-x')} = \frac{-i}{4} \operatorname{sgn}(x-x')$$

$$[\phi(x), \phi(x')] = [\phi_R(x) + \phi_L(x), \phi_R(x') + \phi_L(x')] = [\phi_R(x), \phi_R(x')] + [\phi_L(x), \phi_L(x')]$$

$$+ [\phi_R(x), \phi_L(x')] + [\phi_L(x), \phi_R(x')] = \frac{i}{4} \operatorname{sgn}(x-x')(1-1) + \frac{i}{4}(1-1) = 0$$

similarly $[\theta(x), \theta(x')] = 0$

$$[\phi(x), \theta(x')] = [\phi_R(x) + \phi_L(x), \phi_R(x') - \phi_L(x')] = [\phi_R(x), \phi_R(x')] - [\phi_L(x), \phi_L(x')]$$

$$- [\phi_R(x), \phi_L(x')] + [\phi_L(x), \phi_R(x')] = \frac{i}{2}\left[\operatorname{sgn}(x-x') - 1\right] = -i\,\Theta(x'-x)$$

$$[\phi(x), \partial_x \theta(x')] = -i\,\delta(x'-x) = -i\,\delta(x-x')$$

$$[\phi(x), \partial_{x'}\theta(x')] = -i\delta(x-x'), \implies \partial_x \theta(x) = -\Pi_\phi$$

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

Bosonization identity

$$\psi_R(x) = \frac{1}{\sqrt{L}} e^{i\sqrt{4\pi}\sqrt{\frac{\pi}{L}}\hat{N}_R x} e^{i\sqrt{4\pi}\hat{\varphi}_R^{+}(x)} e^{i\sqrt{4\pi}\varphi_R(x)}$$

$$\implies \boxed{\psi_R(x) = \frac{1}{\sqrt{2\pi a}} e^{i\sqrt{4\pi}\phi_R(x)}}$$

$$\psi_L(x) = \frac{1}{\sqrt{L}} e^{-i\sqrt{4\pi}\sqrt{\frac{\pi}{L}}\hat{N}_L x} e^{-i\sqrt{4\pi}\hat{\varphi}_L^{+}(x)} e^{-i\sqrt{4\pi}\varphi_L(x)}$$

$$\boxed{\psi_L(x) = \frac{1}{\sqrt{2\pi a}} e^{-i\sqrt{4\pi}\phi_L(x)}}$$

if fin multi-component fermions.
- Klein-factors are needed to ensure anti-commutation.

Check anti-commutations:

$$\psi_R(x)\,\psi_R(x') = \frac{1}{2\pi a} e^{i\sqrt{4\pi}\phi_R(x)} e^{i\sqrt{4\pi}\phi_R(x')} = \frac{1}{2\pi a} e^{i\sqrt{4\pi}\phi_R(x')} e^{i\sqrt{4\pi}\phi_R(x)} e^{-4\pi[\phi_R(x), \phi_R(x')]}$$

$$= e^{-4\pi \frac{i}{4}\mathrm{sgn}(x-x')} \psi_R(x')\psi_R(x)$$

$$= e^{-i\pi\,\mathrm{sgn}(x-x')} \psi_R(x')\psi_R(x) = -\psi_R(x')\psi_R(x)$$

Similarly

$$\psi_L(x)\psi_L(x') = -\psi_L(x')\psi_L(x)$$

$$\psi_R(x)\,\psi_L(x') = \frac{1}{2\pi a} e^{i\sqrt{4\pi}\phi_R(x)} e^{-i\sqrt{4\pi}\phi_L(x)} = \frac{1}{2\pi a} e^{-i\sqrt{4\pi}\phi_L(x')} e^{i\sqrt{4\pi}\phi_R(x)} \times e^{4\pi[\phi_R(x), \phi_L(x')]}$$

$$= e^{i\pi} \psi_L(x')\psi_R(x)$$

$$= -\psi_L(x')\psi_R(x)$$

The derivation of bosonization identity is based on the fact that applying fermion operator to the N-particle ground state is equivalent to coherent state of boson operator.

The form in page ④ is already in the normal product

$$e^{i\sqrt{4\pi}\,\varphi_R^{\dagger}(x)}\, e^{i\sqrt{4\pi}\,\varphi_R(x)} = e^{i\sqrt{4\pi}(\varphi_R+\varphi_R^{\dagger})}\, e^{\frac{-1}{2}\cdot 4\pi\,[\varphi_R^{\dagger}(x),\,\varphi_R(x)]}$$

$$= e^{i\sqrt{4\pi}(\varphi_R+\varphi_R^{\dagger})}\, e^{-\frac{1}{2}\ln\frac{2\pi}{L}a} = \left(\frac{L}{2\pi a}\right)^{1/2} e^{i\sqrt{4\pi}(\varphi_R+\varphi_R^{\dagger})}$$

$$\Rightarrow\ \psi_R(x) = \frac{1}{\sqrt{2\pi a}}\, e^{i\sqrt{4\pi}(\varphi_R+\varphi_R^{\dagger}+\sqrt{\frac{\pi}{L}}\hat{N}_R x)} = \frac{1}{\sqrt{2\pi a}}\, e^{i\sqrt{4\pi}\,\phi_R(x)}$$

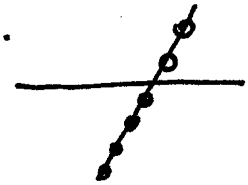
similarly $$e^{-i\sqrt{4\pi}\,\varphi_L^{\dagger}(x)}\, e^{-i\sqrt{4\pi}\,\varphi_L(x)} = e^{-i\sqrt{4\pi}(\varphi_L+\varphi_L^{\dagger})}\, e^{-\frac{1}{2}\cdot 4\pi\,[\varphi_L^{\dagger}(x),\,\varphi_L(x)]}$$

$$= \left(\frac{L}{2\pi a}\right)^{1/2} e^{-i\sqrt{4\pi}(\varphi_L+\varphi_L^{\dagger})}$$

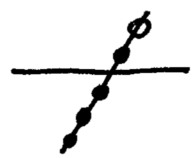
$$\Rightarrow\ \psi_L(x) = \frac{1}{\sqrt{2\pi a}}\, e^{-i\sqrt{4\pi}\,\phi_L(x)}$$

Proof of bosonization identity

we will prove $\psi_\eta |N_L, N_R\rangle_0$ $\eta = R, \text{ or } L$ is a boson coherent state. $|N_R\rangle_0$ means N_R-particle ground state, $N_R = 0$ refers to $k<0$ occupied; $k>0$ empty. 以此为基准.



$|1\rangle_{R0}$ is



, and so on.

check $[b_q, \psi_R(x)] = \alpha_q(x)\, \psi_R(x)$ for $q>0$

$$\Big[\frac{-i}{\sqrt{n_q}} \sum_{k=-\infty}^{+\infty} C^\dagger_{R,k-q} C_{R,k},\ \frac{1}{\sqrt{L}} \sum_{k'=-\infty}^{+\infty} C_{R,k}\, e^{ikx}\Big]$$

$$= \frac{-i}{\sqrt{n_q}} \frac{1}{\sqrt{L}} (-) \sum_k C_{R,k+q}\, e^{i(k+q)x}\, e^{-iqx} = \frac{i\, e^{-iqx}}{\sqrt{n_q}} \psi_R(x) \Rightarrow \alpha_q(x) = \frac{i}{\sqrt{n_q}} e^{-iqx}$$

$$[b_q^\dagger, \psi_R(x)] = \Big[\frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{+\infty} C^\dagger_{R,k+q} C_{R,k},\ \frac{1}{\sqrt{L}} \sum_{k'=-\infty}^{+\infty} C_{R,k}\, e^{ikx}\Big]$$

$$= \frac{i}{\sqrt{n_q}} \frac{1}{\sqrt{L}} (-) \sum_k C_{R,k-q}\, e^{i(k-q)x}\, e^{iqx} = \frac{-i\, e^{iqx}}{\sqrt{n_q}} \psi_R(x)$$

$$\Rightarrow \boxed{\begin{array}{ll} [b_q^\dagger, \psi_R(x)] = \alpha_q^*(x)\, \psi_R(x) & \text{with } \alpha_q(x) = \dfrac{i}{\sqrt{n_q}} e^{-iqx} \\ [b_q, \psi_R(x)] = \alpha_q(x)\, \psi_R(x) & q>0 \end{array}}$$

$$[b_q\ \psi_L(x)] = \left[\frac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{+\infty} C^{\dagger}_{L\,k-q} C_{Lk}\ ,\ \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{+\infty} C_{kL}\, e^{ikx}\right]$$

$$= \frac{-i}{\sqrt{n_q}} \frac{\sum_k C_{L\,k+q}\, e^{i(k+q)x}}{\sqrt{L}}\ e^{-iqx} \quad = \quad \alpha_q(x)\, \psi_L(x)$$

$$[b_q^{\dagger}\ \psi_L(x)] = \left[\frac{-i}{\sqrt{n_q}} \sum_{k=-\infty}^{+\infty} C^{\dagger}_{L\,k+q} C_{L,k}\ \ \frac{1}{\sqrt{L}} \sum_{k=-\infty}^{+\infty} C_{k,L}\, e^{ikx}\right]$$

$$= \frac{i}{\sqrt{n_q}}\ \underbrace{\frac{1}{\sqrt{L}} \sum_k C_{L,k-q}\, e^{i(k-q)x}}\ e^{iqx} = \alpha_q^{*}(x)\ \psi_L(x)$$

$$\boxed{\begin{array}{ll} [b_q^{\dagger}\ ,\ \psi_L(x)] = \alpha_q^{*}(x)\, \psi_L(x) & \text{with } \alpha_q(x) = \dfrac{-i}{\sqrt{n_q}}\, e^{-iqx} \\ [b_q\ \ \psi_L(x)] = \alpha_q(x)\, \psi_L(x) & \text{for } q<0 \end{array}}$$

$$[b_q\ ,\ \psi_R(x)] = \alpha_q(x)\, \psi_R(x) \Rightarrow b_q \psi_R(x) - \psi_R(x) b_q = \alpha_q(x)\, \psi_R(x)$$

$$\Rightarrow \quad b_q\ \underbrace{\psi_R(x)\,|N_R\rangle_0} = \alpha_q(x)\ \underbrace{\psi_R(x)\,|N_R\rangle_0} \qquad \text{for all } q>0$$

$\psi_R(x)|N_R\rangle_0$ is the coherent state for all b_q with $q>0$

$$\Rightarrow \boxed{\psi_R(x)\,|N_R\rangle_0 = \exp\left[\sum_{q>0} \alpha_q(x)\, b_q^{\dagger}\right] \lambda(x)\ F_R |N_R\rangle_0}$$

$\lambda(x)$ is a phase factor to be specified. F_R is a Klein factor representing removing one particle from $|N_R\rangle_0$

for a general case $|N_1, \cdots N_\eta, \cdots N_M\rangle_0$ M- species

$$F_\eta^\dagger |N_1, \cdots N_\eta \cdots N_M\rangle_0 = (-)^{\sum_{i=1}^{\eta-1} N_i} |N_1, \cdots N_\eta+1, \cdots N_M\rangle_0$$

$$F_\eta |N_1 \cdots N_\eta \cdots N_M\rangle_0 = \underbrace{(-)^{\sum_{i=1}^{\eta-1} N_i}}_{\text{phase factor from other species.}} |N_1, \cdots N_\eta-1, \cdots N_M\rangle_0$$

$\Rightarrow \boxed{F_\eta^{-1} = F_\eta^\dagger}$

one species fermion corresponds to one Klein factor.

for excited states $f(b)|N_1 \cdots N_\eta \cdots N_m\rangle_0$

$$F_\eta^\dagger f(b) |N_1 \cdots N_\eta \cdots N_m\rangle_0 = f(b)\ F_\eta^\dagger |N_1, \cdots N_\eta \cdots N_m\rangle_0$$

$$F_\eta\ f(b) |N_1 \cdots N_\eta \cdots N_m\rangle_0 = f(b)\ F_\eta |N_1 \cdots N_\eta \cdots N_m\rangle_0$$

Klein factor , the same configuration of bosonic config, based on the ground state (vaccum) differ by one particle.

Any state can be viewed as some bosonic excitations based on a ground state, and the Klein factor does not change the bosonic configuration, but just change the fermion occupation by one.

for the bosonic sector, F_η really has no effect, but we need to be careful to the minus sign when exchange Klein factors of two different species.

check the equation in the box.

$e^{-B} A e^{B} = A + [A, B]$ if $[A, B]$ commutes with A and B.

or $A e^{B} = e^{B} A + e^{B} [A, B]$

$$b_q e^{\alpha_q(x) b_q^\dagger} = e^{\alpha_q(x) b_q^\dagger} b_q + e^{\alpha_q(x) b_q^\dagger} [b_q, b_q^\dagger] \cdot \alpha_q(x)$$

$$\Rightarrow b_q \underbrace{e^{\alpha(x) b_q^\dagger} \lambda(x) F_R |N_R\rangle_0} = \alpha_q(x) \underbrace{e^{\alpha_q(x) b_q^\dagger} \lambda(x) F_R |N_R\rangle_0}$$

coherent state eigenvalue

Let us rewrite

$$\psi_R(x) |N_R\rangle_0 = e^{\sum_{q>0} \overbrace{\frac{i}{\sqrt{\Omega q}}}^{} e^{-iqx} b_q^\dagger} \lambda(x) F_R |N_R\rangle_0$$

$$\boxed{\psi_R(x) |N_R\rangle_0 = e^{i\sqrt{4\pi}\, \varphi_R^\dagger(x)} \lambda(x) F_R |N_R\rangle_0}$$

left hand side : $\frac{1}{\sqrt{L}} \sum_k e^{ikx} C_{R,k} |N_R\rangle_0$ — an infinite series of single hole excitation.

right hand side: coherent state of boson operator based on N-1 particle vaccum.

Let us check the value of $\lambda(x)$.

$${}_0\langle N_R | F^\dagger \psi_R(x) | N_R\rangle_0 = {}_0\langle N_R | F^\dagger e^{i\sqrt{4\pi}\, \varphi_R^\dagger(x)} \lambda(x) F |N_R\rangle_0$$

$$= \lambda(x)\, {}_0\langle N_R | e^{i\sqrt{4\pi}\, \varphi_R^\dagger(x)} |N_R\rangle_0 = \lambda(x)$$

only "1" contribute

on the other hand, $\psi_R(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} C_{R,k}$

$$\Rightarrow {}_0\langle N_R | F^\dagger \psi_R(x) | N_R\rangle_0 = {}_0\langle N_R - 1 | \psi_R(x) | N_R\rangle_0 = \frac{1}{\sqrt{L}} e^{i \frac{2\pi}{L} :\hat{N}_R: x}$$

only the state just on the Fermi level

$$\Rightarrow \boxed{\lambda(x) = \frac{1}{\sqrt{L}}\, e^{i\sqrt{4\pi}\frac{\sqrt{\pi}}{L}:\hat{N}_R:x}}$$

How about $\psi_R(x)$ acting on an arbitrary state $|\vec{N}\rangle$

$$|N\rangle_R = f(b_q^+)\,|N\rangle_{R0} . \quad (q>0)$$

we need to use the following identies

$$\boxed{\begin{aligned} \psi_R(x)\, f(b_q^+) &= f(b_q^+ - \alpha_q^*(x))\, \psi_R(x) \\ e^{i\sqrt{4\pi}\,\varphi_R(x)} f(b_q^+)\, e^{-i\sqrt{4\pi}\,\varphi_R(x)} &= f(b_q^+ - \alpha_q^*(x)) \end{aligned}}$$

Check: theorem if $[A, B] = DB$ and D commutes with A and B

$$\Rightarrow f(A)B = Bf(A+D). \quad \text{or} \quad Bf(A) = f(A-D)B$$

according to $[\psi_R(x), b_q^+] = -\alpha_q^*(x)\,\psi_R(x)$

$$\Rightarrow \psi_R(x)\, f(b_q^+) = f(b_q^+ - \alpha_q^*(x))\, \psi_R(x) .$$

$e^{-B} f(A)\, e^{B} = f(A+C)$ if $C = [A, B]$ and C commutes with A, B.

$$[b_q^+ \;\; \varphi_R(x)] = \sqrt{\frac{1}{4\pi}}\, \frac{1}{\sqrt{n_q}}\, [b_q^+, b_q]\, e^{+iqx} = \sqrt{\frac{1}{4\pi}}\, \frac{-e^{iqx}}{\sqrt{n_q}}$$

$$[b_q^+, -i\sqrt{4\pi}\,\varphi_R(x)] = +\frac{i e^{iqx}}{\sqrt{n_q}} = -\alpha_q^*(x) = C$$

$$\Rightarrow e^{i\sqrt{4\pi}\,\varphi_R(x)}\, f(b_q^+)\, e^{-i\sqrt{4\pi}\,\varphi_R(x)} = f(b_q^+ - \alpha_q^*(x))$$

$$\Rightarrow \quad \psi_R(x)|N_R\rangle = \psi_R(x)\, f(b_q^\dagger)\,|N_R\rangle_0$$

$$= f(b_q^\dagger - \alpha_q^*(x))\; \psi_R(x)\,|N_R\rangle_0$$

$$= \underbrace{f(b_q^\dagger - \alpha_q^*(x))\;\; e^{i\sqrt{4\pi}\,\varphi_R^\dagger(x)}}_{\text{commute, both only contain } b^\dagger} \lambda(x)\, F_R\,|N_R\rangle_0$$

$$= F_R\, \lambda(x)\; e^{i\sqrt{4\pi}\,\varphi_R^\dagger(x)}\; f(b_q^\dagger - \alpha_q^*(x))\;\; |N_R\rangle_0$$

$$= F_R\, \frac{e^{i\frac{2\pi}{L}:\hat{N}_R:x}}{\sqrt{L}}\; e^{i\sqrt{4\pi}\,\varphi_R^\dagger(x)}\; e^{i\sqrt{4\pi}\,\varphi_R(x)}\; f(b_q^\dagger)\; e^{-i\sqrt{4\pi}\,\varphi_R(x)}\, |N_R\rangle_0$$

$$= \frac{F_R}{\sqrt{L}}\; e^{i\sqrt{4\pi}\,\frac{\sqrt{\pi}}{L}:\hat{N}_R:x}\; e^{i\sqrt{4\pi}\,\varphi_R^\dagger(x)}\; e^{i\sqrt{4\pi}\,\varphi_R(x)}\,|N_R\rangle$$

$$\Rightarrow \quad \boxed{\psi_R(x) = \frac{F_R}{\sqrt{L}}\; e^{i\sqrt{4\pi}\,\frac{\sqrt{\pi}}{L}N_R x}\;\; e^{i\sqrt{4\pi}\,\varphi_R^\dagger(x)}\; e^{i\sqrt{4\pi}\,\varphi_R(x)}}$$

simplify to

$$\frac{F_R}{\sqrt{2\pi a}}\; e^{i\sqrt{4\pi}\,\phi_R(x)}$$

where $\phi_R(x) = \varphi_R^\dagger(x) + \varphi_R(x) + \frac{\sqrt{\pi}}{L} N_R\, x$

~~OPE method~~ with respect to the free particle vaccum

$$\rho_R(x) = :\psi_R^\dagger(x)\psi_R(x): = \lim_{\epsilon\to 0}\lim_{a\to 0}\left(\psi_R^\dagger(x+\epsilon)\psi_R(x) - \langle|\psi_R^\dagger(x+\epsilon)\psi_R(x)|\rangle\right)$$

$$\psi_R^\dagger(x+\epsilon)\psi_R(x) = \frac{1}{2\pi a}\, e^{-i\sqrt{4\pi}\,\phi_R(x+\epsilon)}\, e^{i\sqrt{4\pi}\,\phi_R(x)}$$

$$e^A e^B = :e^{A+B}:\; e^{\langle AB + \frac{A^2+B^2}{2}\rangle}$$

$$\Rightarrow e^{-i\sqrt{4\pi}\phi_R(x+\epsilon)}\, e^{i\sqrt{4\pi}\phi_R(x)} = :e^{-i\sqrt{4\pi}(\phi_R(x+\epsilon)-\phi_R(x))}:\; e^{4\pi\langle\phi_R(x+\epsilon)\phi_R(x) - \phi_R^2(0)\rangle}$$

$$\langle|\phi_R(x+\epsilon)\phi_R(x)|\rangle = \langle|\varphi_R(x+\epsilon),\ \varphi_R^\dagger(x)|\rangle = \langle|[\varphi_R(x+\epsilon),\ \varphi_R^\dagger(x)]|\rangle$$

$$\to [\varphi_R(x),\ \varphi_R^\dagger(x')] = \frac{-1}{4\pi}\ln\frac{2\pi}{L}(a - i(x-x'))$$

$$\Rightarrow e^{4\pi\langle\phi_R(x+\epsilon)\phi_R(x) - \phi_R^2(0)\rangle} = e^{-\left[\ln\frac{2\pi}{L}(a-i\epsilon) - \ln\frac{2\pi}{L}a\right]}$$

$$= e^{-\ln\frac{a-i\epsilon}{a}} = \frac{a}{a-i\epsilon}$$

$$\Rightarrow :\psi_R^\dagger(x+\epsilon)\psi_R(x): = \lim_{\epsilon\to 0}\lim_{a\to 0}\frac{1}{2\pi a}\left[e^{-i\sqrt{4\pi}\phi_R(x+\epsilon)}\, e^{i\sqrt{4\pi}\phi_R(x)} - \langle|e^{-i\sqrt{4\pi}\phi_R(x+\epsilon)}e^{i\sqrt{4\pi}\phi_R(x)}|\rangle\right]$$

$$e^{-i\sqrt{4\pi}\phi_R(x+\epsilon)}\, e^{i\sqrt{4\pi}\phi_R(x)} = :e^{-i\sqrt{4\pi}\,\epsilon\,\partial_x\phi_R}:\ \frac{a}{a-i\epsilon}$$

$$\Rightarrow :\psi_R^\dagger(x+\epsilon)\psi_R(x): = \lim_{\epsilon\to 0}\lim_{a\to 0}\frac{1}{2\pi a}\left[:e^{-i\sqrt{4\pi}\,\epsilon\,\partial_x\phi_R}: - 1\right]\frac{a}{a-i\epsilon}$$

$$= \lim_{\epsilon\to 0}\lim_{a\to 0}\frac{1}{2\pi a}\left(-i\sqrt{4\pi}\,\epsilon\,\partial_x\phi_R\right)\frac{a}{a-i\epsilon} = \sqrt{\frac{1}{\pi}}\,\partial_x\phi_R(x)$$

operator normal product

$$A = \alpha a + \alpha' a^\dagger \qquad B = \beta a + \beta' a^\dagger$$

if $[A,B]$ commutes with A, B

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} = e^B e^A e^{[A,B]}$$

$$:e^A::e^B: = e^{\alpha' a^\dagger} e^{\alpha a} e^{\beta' a^\dagger} e^{\beta a} = e^{\alpha' a^\dagger} e^{\beta' a^\dagger} e^{\alpha a} e^{\beta a} e^{[\alpha a, \beta' a^\dagger]}$$

$$= :e^{A+B}: e^{\langle 0|AB|0\rangle}$$

$$e^A e^B = e^{\alpha' a^\dagger + \alpha a} e^{\beta a^\dagger + \beta a} = e^{\alpha' a^\dagger} e^{\alpha a} e^{\beta' a^\dagger} e^{\beta a} \cdot e^{-\frac{1}{2}[\alpha' a^\dagger, \alpha a] - \frac{1}{2}[\beta' a^\dagger, \beta a]}$$

$$= :e^A::e^B: e^{\frac{1}{2}\alpha\alpha' + \frac{1}{2}\beta\beta'}$$

$$= :e^{A+B}: e^{\langle 0| AB + \frac{A^2}{2} + \frac{B^2}{2} |0\rangle}$$

$$\rho_L(x) = :\psi_L^\dagger(x)\psi_L(x): = \lim_{\epsilon\to0}\lim_{a\to0} \psi_L^\dagger(x+\epsilon)\psi_L(x) - \langle|\psi_L^\dagger(x+\epsilon)\psi_L(x)|\rangle$$

$$\psi_L^\dagger(x+\epsilon)\psi_L(x) = \frac{1}{2\pi a} e^{i\sqrt{4\pi}\phi_L(x+\epsilon)} e^{-i\sqrt{4\pi}\phi_L(x)}$$

$$= \frac{1}{2\pi a} :e^{i\sqrt{4\pi}\,\epsilon\,\partial_x\phi_L}: \; e^{4\pi\langle|\phi_L(x+\epsilon)\phi_L(x) - \phi_L^2(0)|\rangle}$$

$$[\varphi_L(x), \varphi_L(x')] = \frac{-1}{4\pi} \ln \frac{2\pi}{L}(a + i(x-x'))$$

$$\langle|\phi_L(x+\epsilon)\phi_L(x) - \phi_L^2(0)|\rangle = \frac{-1}{4\pi} \ln \frac{a+i\epsilon}{a}$$

$$\Rightarrow :\psi_L^\dagger(x)\psi_L(x): = \frac{1}{2\pi a} :\underbrace{e^{i\sqrt{4\pi}\,\epsilon\,\partial_x\phi_L}-1}_{i\sqrt{4\pi}\,\epsilon\,\partial_x\phi_L}: \left(\frac{a}{a+i\epsilon}\right) = \sqrt{\frac{1}{\pi}}\,\partial_x\phi_L$$

check: kinetic energy

$$:\psi_R^\dagger\, i\partial_x\,\psi_R: = \frac{1}{\epsilon}\Big\{\psi_R^\dagger(x)\big[\psi_R(x+\tfrac{\epsilon}{2}) - \psi_R(x-\tfrac{\epsilon}{2})\big] - \langle|\psi_R^\dagger(x)\big(\psi_R(x+\tfrac{\epsilon}{2}) - \psi_R(x-\tfrac{\epsilon}{2})\big)|\rangle\Big\}$$

$$= \lim_{\epsilon\to0}\lim_{a\to0} \frac{1}{\epsilon}\Big\{\psi_R^\dagger(x)\psi_R(x+\tfrac{\epsilon}{2}) - \langle|\psi_R^\dagger(x)\psi_R(x+\tfrac{\epsilon}{2})|\rangle - \Big(\psi_R^\dagger(x)\psi_R(x-\tfrac{\epsilon}{2}) - \langle|\psi_R^\dagger(x)\psi_R(x-\tfrac{\epsilon}{2})|\rangle\Big\}$$

$$\psi_R^\dagger(x)\psi_R(x+\tfrac{\epsilon}{2}) - \langle|\psi_R^\dagger(x)\psi_R(x+\tfrac{\epsilon}{2})|\rangle = \frac{1}{2\pi a}\Big[:e^{+i\sqrt{4\pi}\frac{\epsilon}{2}\partial_x\phi_R}: - 1\Big]\frac{a}{a+i\epsilon/2}$$

$$= \frac{1}{2\pi}\frac{1}{+i\epsilon/2}\Big[\frac{i\sqrt{4\pi}}{2}\epsilon:\partial_x\phi_R: + \frac{1}{2}(i\sqrt{\pi})^2\epsilon^2:(\partial_x\phi_R)^2:\Big]$$

$$= \frac{1}{2\pi}\frac{1}{+i\epsilon/2}\Big[\cancel{\ \ } i\sqrt{\pi}\,\epsilon:\partial_x\phi_R: - \frac{\pi}{2}\epsilon^2(\partial_x\phi_R)^2\Big]$$

$$\psi_R^\dagger(x)\psi_R(x-\tfrac{\epsilon}{2}) - \langle 1 \cdots 1\rangle = \frac{1}{2\pi}\frac{1}{-i\epsilon/2}\left[-i\sqrt{\pi}\epsilon :\partial_x\phi_L: - \frac{\pi}{2}\epsilon^2(\partial_x\phi)_R^2\right.$$

$$\Rightarrow \lim_{\epsilon\to 0}\frac{1}{\epsilon}\frac{1}{2\pi}\frac{1}{i\epsilon/2}\cdot\pi\,\epsilon^2(\partial_x\phi_R)^2 = \frac{1}{i}(\partial_x\phi_R)^2$$

$$\Rightarrow :\psi_R^\dagger\, i\partial_x\psi_R: \;=\; :(\partial_x\phi_R)^2:$$

Similarly $\psi_L^\dagger(-i\partial_x)\psi_L = :(\partial_x\phi_L)^2:$

Spinless

$$* \quad \boxed{\begin{aligned} \rho(x) &= \rho_R + \rho_L = \sqrt{\tfrac{1}{\pi}}\,\partial_x(\phi_R+\phi_L) = \sqrt{\tfrac{1}{\pi}}\,\partial_x\phi \\ j(x) &= v(\rho_R-\rho_L) = v\sqrt{\tfrac{1}{\pi}}\,\partial_x(\phi_R-\phi_L) = v\sqrt{\tfrac{1}{\pi}}\,\partial_x\theta \end{aligned}}$$

$$N = :\psi_R^\dagger\psi_L: = \frac{1}{2\pi a}:e^{-i\sqrt{4\pi}\phi_R}\,e^{-i\sqrt{4\pi}\phi_L}:$$

$$= \frac{1}{2\pi a}:e^{-i\sqrt{4\pi}\phi}:\; e^{-4\pi[\phi_R(x),\phi_L(x)]/2} \quad \leftarrow e^{-2\pi\cdot\frac{i}{4}}$$

$$\boxed{\begin{aligned} N &= \frac{-i}{2\pi a}e^{i\sqrt{4\pi}\phi} \\ N^\dagger &= :\psi_L^\dagger\psi_R: = \frac{i}{2\pi a}e^{-i\sqrt{4\pi}\phi} \end{aligned}}$$

$$e^{2\pi\frac{i}{4}}$$

$$\Delta = :\psi_R^\dagger\psi_L^\dagger: = \frac{1}{2\pi a}e^{i\sqrt{4\pi}\phi_R}\,e^{i\sqrt{4\pi}\phi_L} = \frac{1}{2\pi a}e^{-i\sqrt{4\pi}\theta}\,e^{4\pi[\phi_R,\phi_L]/2}$$

$$= \frac{i}{2\pi a}e^{-i\sqrt{4\pi}\theta}$$

$$\boxed{\begin{aligned} \Delta &= \frac{i}{2\pi a}e^{-i\sqrt{4\pi}\theta} \\ \Delta^\dagger &= \frac{-i}{2\pi a}e^{i\sqrt{4\pi}\theta} \end{aligned}}$$