Lect 2: Bethe ansatz (II) – many magons

⊛ Bethe ansatz Eq for many magnon states

$$\psi(x_1, \cdots x_M) = \sum_P A_P \, e^{i \sum_{l=1}^{M} k_{P_l} x_l} \qquad (x_1 < x_2 < \cdots x_M)$$

Again if nune of $x_1, \cdots x_M$ are adjacent to each other, the $H\psi = E\psi$ will be

$$\frac{J}{2} \sum_{l=1}^{M} [\psi(x_1, \cdots x_l+1, \cdots x_M) + \psi(x_1, \cdots x_l-1, \cdots x_M)]$$

$$-JM\Delta \, \psi(x_1, \cdots x_M) = E\psi(x_1, \cdots x_M). \qquad (*)$$

If there exists one $\underbrace{\text{pair } (x_j, x_{j+1} = x_j+1)}_{\text{of neighbors}}$, and others are not.

then $\frac{J}{2} \sum_l' \psi(x_1, \cdots x_l+1, x_M) + \psi(x_1 \cdots x_l-1, \cdots x_M)] - J(M-1)\Delta\psi(x_1, \cdots x_M)$

$$= E\psi(x_1, \cdots x_M) \qquad (**)$$

where $\sum_l'$ excludes the term $\psi(\cdots x_j+1, x_{j+1}, \cdots) + \psi(\cdots x_j, x_{j+1}-1, \cdots)$

$$= \psi(\cdots x_{j+1}, x_{j+1}, \cdots) + \psi(\cdots x_j, x_j, \cdots)$$

Hence, the difference between (*) and (**) can be fixed by requiring

$$\boxed{\frac{J}{2}[\psi(\cdots \underset{\substack{\uparrow \\ j}}{x_{j+1}}, \underset{\substack{\uparrow \\ j+1}}{x_{j+1}}, \cdots) + \psi(\cdots \underset{\substack{\uparrow \\ j}}{x_j}, \underset{\substack{\uparrow \\ j}}{x_j} \cdots)] - J\Delta\psi(x_1, \cdots x_m) = 0}$$

Once this condition is met, we can also reconsile the case with 3-magnun touching each other.

Assume that three are three "$\downarrow$" are neighbours $\underset{j-1}{\downarrow}\ \underset{j}{\downarrow}\ \underset{j+1}{\downarrow}$

All other "$\downarrow$"'s are not adjacent to each other.

Then $\sum\limits_{\ell} \cdots$ does not include the following "non-physical" terms

of
$$\psi(\cdots \underset{\overset{\uparrow}{j-1}}{x_{j-1}+1},\ \underset{\overset{\uparrow}{j}}{x_j},\ \underset{\overset{\uparrow}{j+1}}{x_{j+1}} \cdots) + \psi(\cdots x_{j-1},\ x_j-1,\ x_{j+1}\cdots)$$

$$+\psi(\cdots x_{j-1},\ x_j+1,\ x_{j+1},\cdots) + \psi(\cdots x_{j-1},\ x_j,\ x_{j+1}-1,\cdots)$$

$$= \psi(\cdots \underset{\overset{\uparrow}{j-1}}{x_j}\ \underset{\overset{\uparrow}{j}}{x_j} \cdots) + \psi(\cdots \underset{\overset{\uparrow}{j-1}}{x_{j-1}},\ \underset{\overset{\uparrow}{j}}{x_{j-1}} \cdots) + \psi(\cdots \underset{\overset{\uparrow}{j}}{x_{j+1}},\ \underset{\overset{\uparrow}{j+1}}{x_{j+1}} \cdots)$$

$$+\psi(\cdots \underset{\overset{\uparrow}{j-1}}{x_j}\ \underset{\overset{\uparrow}{j}}{x_j} \cdots)$$

$\downarrow\downarrow\downarrow$ 4 domain walls missing.

now $n(x_1 \cdots x_M) = 2M-4$

The eigen equation $H\psi = E\psi$ goes to

$$\frac{J}{2}\sum_{\ell}{}' \left\{\psi(\cdots \underset{\overset{\uparrow}{\ell}}{x_\ell+1}\cdots) + \psi(\cdots \underset{\overset{\uparrow}{\ell}}{x_\ell-1}\cdots)\right\} - J(m-2)\Delta\psi(x_1\cdots x_m)$$

(unphysial terms excluded)

$$= E\,\psi(x_1\cdots x_m)$$

③

the Bethe ansatz solution $\psi(x_1 \cdots x_n) = \sum_P A_P \, e^{i \sum_{l=1}^{M} K_{P_l} X_l}$

all the K's are different

satisfies

$$\frac{J}{2} \sum_l \{\psi(x_1 \cdots x_l + 1, \cdots x_m) + \psi(x_1 \cdots x_l - 1, \cdots x_m)\}$$

see below

$$-J m \Delta \psi(x_1 \cdots x_m) = E \psi(x_1 \cdots x_m)$$

$\Rightarrow$ The difference reads

$$\frac{J}{2}\left[2\psi(\cdots x_j, x_j \cdots) + \psi(\cdots x_{j-1} x_{j-1} \cdots) + \psi(\cdots x_{j+1} x_{j+1} \cdots)\right]$$

$$- 2J \Delta \psi(x_1 \cdots x_m) = 0$$

which is consistent with if we impose the condition

$$\frac{J}{2}(\psi(\cdots \underset{j-1}{\overset{\uparrow}{x_{j-1}}} \underset{j}{\overset{\uparrow}{x_{j-1}}} \cdots) + \psi(\cdots \underset{j-1}{\overset{\uparrow}{x_j}} \underset{j}{\overset{\uparrow}{x_j}} \cdots)] = J \Delta \psi(x_1 \cdots x_M)$$

$$\& \quad \frac{J}{2}(\psi(\cdots \underset{j}{\overset{\uparrow}{x_j}} \underset{j+1}{\overset{\uparrow}{x_j}} \cdots] + \psi(\cdots \underset{j}{\overset{\uparrow}{x_{j+1}}} \underset{j+1}{\overset{\uparrow}{x_{j+1}}} \cdots)] = J \Delta \psi(x_1 \cdots x_m)$$

This procedure can be generalized, Bethe ansatz solution satisfy the $H\psi = E\psi$, if the condition of

$$\boxed{\frac{1}{2}(\psi(\cdots \underset{j}{\overset{\uparrow}{x_j}} \underset{j+1}{\overset{\uparrow}{x_j}} \cdots) + \psi(\cdots \underset{j}{\overset{\uparrow}{x_{j+1}}} \underset{j+1}{\overset{\uparrow}{x_{j+1}}} \cdots)] = \Delta \psi(x_1 \cdots x_m)}$$

is satified for arbitary config of $x_1 \cdots x_M$, and arbitary j.

HW: prove the general configuration of $x_1 < x_2 < \cdots x_M$, that the condition

$$\frac{J}{2}\Big[\psi(x \cdots \underset{j}{x_j}, \underset{j+1}{x_j} \cdots) + \psi(\cdots \underset{j}{x_{j+1}}, \underset{j+1}{x_{j+1}} \cdots)\Big] - J\Delta\,\psi(x_1 \cdots x_M) = 0 \quad \text{for } x_{j+1} = x_j + 1$$

is compatible with the BA solution.

Now we detemine the scattering amplitude A_P.

Set $(j, j+1) = (1, 2)$, we have $\psi(x_1, x_1) + \psi(x_1+1, x_1+1) = 2\Delta\,\psi(x_1, x_1+1, \cdots x_M)$

$$\Rightarrow \sum_P A_P\left(e^{ik_{P_1}x_1 + ik_{P_2}x_1} + e^{ik_{P_1}(x_1+1) + ik_{P_2}(x_1+1)}\right) \otimes e^{i\sum_{j>2} k_{P_j} x_j}$$

$$= 2\Delta \sum_P A_P\, e^{ik_{P_1}x_1 + ik_{P_2}(x_1+1)} \otimes e^{i\sum_{j>2} k_{P_j} x_j}$$

$$\Rightarrow \sum_P \left\{A_P\left(e^{ik_{P_1} + ik_{P_2}} - 2\Delta e^{ik_{P_2}} + 1\right) \cdot e^{i\sum_{j>2} k_{P_j} x_j}\right\} = 0$$

we organize A_P into two classes $P = (P_1, P_2 \cdots P_M)$

and $P' = (P_2, P_1, \cdots P_M) = (1,2)\,P$

$$\Rightarrow \sum_P \left\{A_P\left(e^{ik_{P_1}+ik_{P_2}} - 2\Delta e^{ik_{P_2}} + 1\right) + A_{P'}\left(e^{i(k_{P_2}+k_{P_1})} - 2\Delta e^{ik_{P_1}} + 1\right)\right\}$$

$$\cdot\, e^{i\sum_{j>2} k_{P_j} x_j} = 0$$

hence

$$\frac{A_{P'}}{A_P} = -\frac{e^{i(k_{P_1}+k_{P_2})} - 2\Delta e^{ik_{P_2}} + 1}{e^{i(k_{P_1}+k_{P_2})} - 2\Delta e^{ik_{P_1}} + 1}$$

Generally, if two permutation can be conneteel by an exange of neighbouring pair j and $j+1$, i.e.

$$k_{p_1}\ k_{p_2} \cdots = \cdots \overset{j}{k}\ \overset{j+1}{k'} \cdots$$

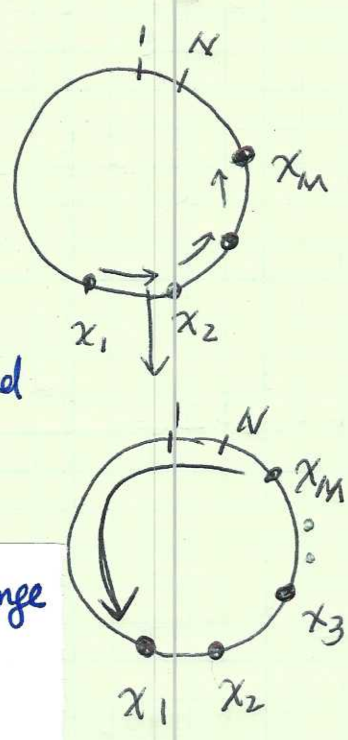
$$k_{p'_1}\ k_{p'_2} \cdots = \cdots k'\ k \cdots$$

$$\Rightarrow \boxed{\frac{A_{p'}}{A_p} \equiv -e^{i\Theta(k',k)} = -\frac{e^{i(k+k')}-2\Delta e^{ik'}+1}{e^{i(k+k')}-2\Delta e^{ik}+1}}$$

* Periodical boundry condition

Set $X_1 \to X_1 + N$, and let all other X_j unmoveel

$$\psi(x_1, \cdots x_M) = \psi(x_2, \cdots x_M, x_1+N).$$



Let us pick up the term $A_{12\cdots M}\, e^{ik_1x_1+\cdots k_Mx_M}$, and let particle 1 with momentum k_1 go through the ring to collide with particles 2, 3, $\cdots$ M and return to x_1+N.

① as $x_1 \to x_1 + N$, the plane wave k_1 phase change e^{ik_1N}

② first collision $0<r_1<x_1 \longrightarrow x_1 < r < x_2$

$$\underset{\cdots\ \psi(r_1\, x_1\, x_2 \cdots)}{A_{12\cdots M}\, e^{i(k_1r_1+k_2r_2+\cdots k_Mr_M)}} \xrightarrow[(4.2)]{} \underset{\psi(x_1\, r_1\, x_2 \cdots)}{A_{21\cdots M}\, e^{i(k_2r_2+k_1x_1+\cdots k_Mr_M)}}$$

③ The n-collision $\quad x_n < r_1 < x_{n+1} \longrightarrow x_{n+1} < r_1 < x_{n+2}$.

$$\psi(x_2, \cdots x_n, r_1, x_{n+1} \cdots) \longrightarrow \psi(x_2, \cdots x_{n+1}, r_1, x_{n+2} \cdots)$$

$$A_{23\cdots n1\,n+1\cdots}\, e^{ik_2x_2 \cdots k_nr_n + k_1r_1 + \cdots} \longrightarrow A_{23\cdots n+1\,1}\, e^{ik_2x_2 + \cdots k_1r_1 + \cdots}$$

hence we have a series of $M-1$ collisions, the phase shifts.

$$A_{12\cdots M} \longrightarrow A_{21\cdots M} \longrightarrow A_{231\cdots M} \longrightarrow \cdots \quad A_{23\cdots M1}$$

we have $e^{ik_1N} \dfrac{A_{23\cdots M1}}{A_{12\cdots M}} = 1$, or

$$\boxed{e^{ik_1N} \frac{A_{21\cdots M}}{A_{12\cdots M}} \cdot \frac{A_{231\cdots M}}{A_{21\cdots M}} \cdots \frac{A_{23\cdots M1}}{A_{23\cdots 1M}} = 1}$$

since $\dfrac{A_{23\cdots n1\,n+1\,M}}{A_{23\cdots 1,n,n+1,M}} = -e^{i\Theta(k_n,k_1)} = -\dfrac{e^{i(k_n+k_1)} - 2\Delta e^{ik_n} + 1}{e^{i(k_n+k_1)} - 2\Delta e^{ik_1} + 1}$

$$\Rightarrow \boxed{e^{ik_1N} (-)^{M-1} e^{i\sum_{l=1}^{M} \Theta(k_l, k_1)} = 1} \Leftarrow \Theta(k_1, k_1) = 0$$

Basically, we have decompose a many-body scattering amplitude i.e. a rotation into a product of 2-particle exchange.

- Similarly, we can pick up the $A_{21\cdots M}\, e^{ik_2r_1 + k_1\cancel{x_2} + k_3x_3 + \cdots}$ in the wavefunction $\psi(r_1, x_2, \cdots x_M)$

then as r_1 runs from $x_1 \to x_2 \to \cdots \to x_M \to x_1+N$, we have

$$\left.\begin{array}{lll} \psi(r_1, x_2\, x_3 \cdots) & \to & \psi(x_2, r_1, x_3 \cdots) \\ A_{213\cdots M}\, e^{i k_2 r_1 + k_1 x_2 + \cdots k_M r_M} & \longrightarrow & A_{123\cdots M}\, e^{i k_1 x_2 + k_2 r_1 + k_3 x_3 + \cdots} \end{array}\right\}$$

$$\longrightarrow \psi(x_2, x_3, r_1 \cdots) \quad \stackrel{\cdots}{\longrightarrow} \quad \psi(x_2, x_3 \cdots x_N, r_1)$$

$$\underbrace{A_{132\cdots M}}\, e^{i(k_1 x_2 + k_3 x_3 + k_2 r_1 + \cdots)} \qquad A_{13\cdots M2}\, e^{i(k_1 x_2 + k_3 x_3 \cdots k_2 r_1)}$$

same sequence of momentum

$$\Rightarrow \quad e^{ik_2 N} \frac{A_{123\cdots M}}{A_{213\cdots M}} \frac{A_{132\cdots M}}{A_{123\cdots M}} \cdots \frac{A_{13\cdots M2}}{A_{13\cdots 2M}} = 1$$

since $$\frac{A_{13\cdots n2\cdots}}{A_{13\cdots 2n\cdots}} = -e^{i\Theta(k_n, k_2)} = -\frac{e^{i(k_n+k_2)} - 2\Delta e^{ik_n} + 1}{e^{i(k_n+k_1)} - 2\Delta e^{ik_1} + 1}$$

$$\Rightarrow \quad e^{ik_2 N} (-)^{M-1} e^{i\sum_{\ell=1}^{M} \Theta(k_\ell, k_2)} = 1. \quad \leftarrow \text{Note } \Theta(k_\ell, k_2) = -\Theta(k_2, k_\ell)$$

Hence in general we have the following Bethe Ansatz Eq:

$$e^{ik_j N} = (-)^{M-1} e^{i\sum_{\ell=1}^{M} \Theta(k_j, k_\ell)} \quad \text{for } j = 1, 2, \cdots M$$

$$\Rightarrow \quad k_j N = 2\pi Q_j + \sum_{\ell=1}^{M} \Theta(k_j, k_\ell) \quad \text{with}$$

Q_j = integer if M is odd

half integer if M is even

and $$\Theta(k_j, k_\ell) = \frac{e^{i(k_j+k_\ell)} - 2\Delta e^{ik_j} + 1}{e^{i(k_j+k_\ell)} - 2\Delta e^{ik_\ell} + 1}$$

HW : Directly derive the BA equation from Periodical boundary condition :

Set $x_1 \to x_1 + N$, and let all other x_j, unmoved

$$\psi(x_1 \cdots x_m) = \psi(x_2, \cdots x_m, x_1+N)$$

$\uparrow$ x_1+N is the largest index.

$$\Rightarrow \sum_P A_P e^{ik_{P_1}x_1 + \cdots ik_{P_m}x_m} = \sum_P A_P e^{ik_{P_1}x_2 + k_{P_2}x_3 + \cdots ik_{P_M}x_1 + ik_{P_M}N}$$

define $P' = (P_2, P_3 \cdots P_m, P_1)$

$$\Rightarrow RHS = \sum_{P'} A_{P'} e^{ik_{P_2}x_2 + ik_{P_3}x_3 + \cdots ik_{P_1}x_1} e^{ik_{P_1}\cdot N}$$

$$\Rightarrow \boxed{A_{P'} e^{ik_{P_1}N} = A_P \quad \text{where } \left.\begin{matrix} P = (P_1, P_2, \cdots P_N) \\ P' = (P_2, P_3 \cdots P_1) \end{matrix}\right\} \text{rotation}}$$

on the other hand $A_{P'} = -e^{i\Theta(k_{P_{j+1}}, k_{P_j})} A_P$ if $\begin{matrix} P = (P_1 \cdots P_j P_{j+1} \cdots) \\ P' = (P_1 \cdots P_{j+1} P_j \cdots) \end{matrix}$

$$\Rightarrow A_{P'} = A_{P_2 P_3 \cdots P_m P_1} = (-) e^{i\Theta(k_{P_m}, k_{P_1})} A_{P_2 P_3 \cdots P_1 P_m}$$

$$= (-)^2 e^{i\Theta(k_{P_m}, k_{P_1}) + i\Theta(k_{P_{m-1}}, k_{P_1})} A_{P_2 P_3 \cdots P_1 P_{m-1} P_m}$$

$$= (-)^{m-1} e^{i\sum_{l=2,\cdots m} \Theta(k_{P_l}, k_{P_1})} A_{P_1 \cdots P_m}$$

$\leftarrow$ $\Theta(k_{P_1}, k_{P_1}) = 0$

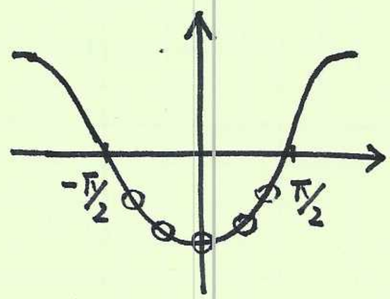
$$\Rightarrow \boxed{e^{ik_{P_1}N} = (-)^{m-1} e^{-i\sum_{l=1}^{m} \Theta(k_{P_l}, k_{P_1})}}$$

§ XY limit, and set $J \to -J$ (ferro XY model, $\Delta = 0$)

$$e^{ik_j N} = (-)^{M-1} \Rightarrow k_j = \frac{2\pi}{N} Q_j$$

Q_j: integer for odd M but half-integer for even M

$$E = -J \sum_{j=1}^{M} \cos k_j$$



For the ground state

$$Q_j = -\frac{M-1}{2}, -\frac{M-3}{2}, \ldots \frac{M-1}{2}$$

If M is odd, $k=0$ is included

If M is even, $k=0$ is not included.

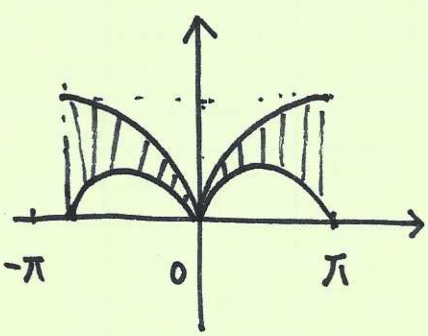
① Excitation spectrum: p-h (particle # conserved)

$$E(q) = -J[\cos(k+q) - \cos k] = 2J \sin(k + q/2) \sin q/2$$

The upper bound at $k = \frac{\pi}{2} - q/2 \Rightarrow E_u(q) = 2J \sin q/2$

The lower bound $\frac{\pi}{2} - q < k < \pi/2 \Rightarrow \frac{\pi}{2} - q/2 < k + q/2 < \frac{\pi}{2} + \frac{q}{2}$

$$\sin_{\min}(k + q/2) = \sin\left(\frac{\pi}{2} \mp \frac{q}{2}\right) = \cos\frac{q}{2} \Rightarrow E_L(q) = J|\sin q|.$$



② Consider S_+, then the ground state change particle number

$$\sum_{Q_j=-\frac{M-1}{2}}^{\frac{M-1}{2}} \cos\frac{2\pi}{N}Q_j = \sum_{Q_j=-\frac{M-1}{2}}^{\frac{M-1}{2}} e^{i\frac{2\pi}{N}Q_j}$$

$$= \frac{e^{-i\frac{2\pi}{N}\frac{M-1}{2}} - e^{i\frac{2\pi}{N}\frac{M+1}{2}}}{1-e^{i\frac{2\pi}{N}}} = \frac{e^{-i\frac{2\pi}{N}\frac{M}{2}} - e^{i\frac{2\pi}{N}\frac{M}{2}}}{e^{-i\pi/N}-e^{i\pi/N}}$$

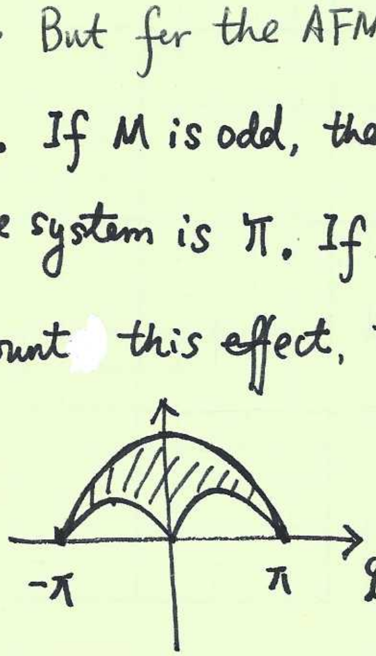
$$= \frac{\sin\left(\frac{M}{N}\pi\right)}{\sin(\pi/N)}$$

$\Rightarrow E_G(M) = -J\sin\frac{M}{N}\pi \,/\sin\frac{\pi}{N}$. At zero field, $M=\frac{N}{2}$, $E_G(M)$ reaches the lowest value. But as M changes from $\frac{N}{2}\to\frac{N}{2}\pm1$, the $E_G(\frac{N}{2}\pm1)$

$= -J\sin(\frac{1}{2}\pm\frac{1}{N})\pi\,/\sin\frac{\pi}{N} \Rightarrow E_G(\frac{N}{2}\pm1)-E_G(\frac{N}{2}) = \frac{J}{\sin\frac{\pi}{N}}\left[1-\cos\frac{\pi}{N}\right]$

$$\longrightarrow \frac{J}{2}\left(\frac{\pi}{N}\right) \to 0 \text{ as } N\to\infty.$$

So the extra p-h excitation based on $|\frac{N}{2}\pm1\rangle_G$ has the same shape as before. But for the AFM XY chain, the spectra minimal is located at $k=\pi$. If M is odd, the state of $k=\pi$ is occupied $\Rightarrow$ the total momenta of the system is π. If M is even, $\Rightarrow$ the total momenta is zero. To count this effect, the $S_\pm$ spectra is shifted by π.



the upper boundary $2J\cos\frac{q}{2}$

the lower one $J|\sin q|$.

② consider the case in the external field h with polarization

$$H = -J\sum_i (S_{x,i} S_{x,i+1} + S_{y,i} S_{y,i+1}) - h \sum_i S_{z,i}$$

For the case that the $|G\rangle$ has M down spin : $M < N/2$, then the ~~total~~ $S_z = \{(N-M) - M\}/2 = \frac{N}{2} - M$. The condition between M and h is $\boxed{E_G(M-1) - h < E_G(M) < E_G(M+1) + h}$

or in the continuum limit $\boxed{h = -\frac{\partial E_G}{\partial M} = +J \frac{\cos \frac{M}{N}\pi}{\sin \pi/N} \frac{\pi}{N}} > 0$

for $M < \frac{N}{2}$.

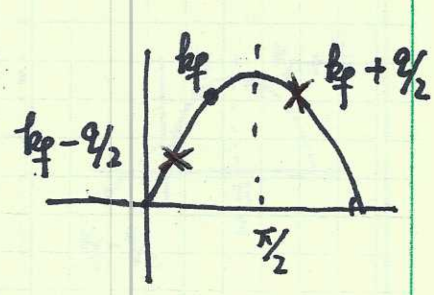
first consider particle-number conserved excitations

$$\boxed{k_f = \frac{2\pi}{N}\frac{M-1}{2} \sim \frac{\pi}{N} M} < \frac{\pi}{2}$$

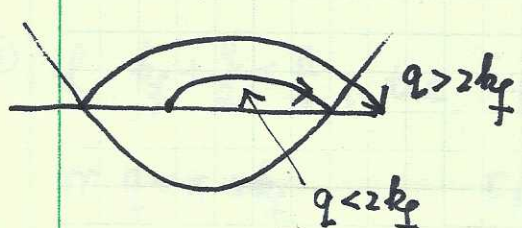
$$E(q) = J[\cos k - \cos(k+q)] \quad \text{for } k_f - q < k < k_f \Rightarrow k_f - q/2 < k + q/2 < k_f + q/2$$

① Since $k_f < \pi/2$, the lower bound of $\sin(k + q/2)$ is taken at $k = k_f - q$

$$E_{lower}(q) = J[-\cos k_f + \cos(k_f - q)]$$
$$= 2J \sin(k_f - q/2) \sin q/2 \quad \text{for } q < 2k_f$$



if $q > 2k_f$ $\quad E_{low}(q) = J[-\cos(-k_f + q) + \cos k_f]$



$\Rightarrow$ in total, we have

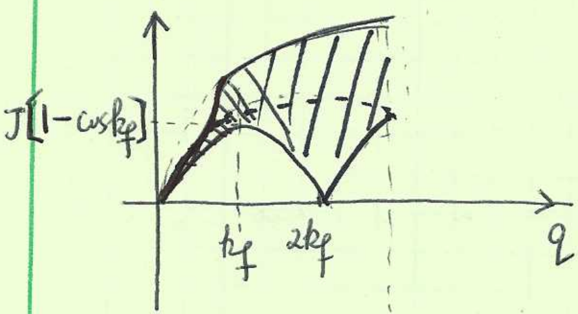
$$E_{low}(q) = 2J\,|\sin(k_f - q/2)|\, \sin\frac{q}{2}$$

② the upper bound. If $k_f + q/2 > \frac{\pi}{2}$, or $q > \pi - 2k_f$

$$E_{up}(q) = 2J \sin q/2 .$$

If $k_f + q/2 < \pi/2$ or $q < \pi - 2k_f$, $\Rightarrow E_{up} = J[\cos(k_f) - \cos(k_f + q)]$

$$= 2J \sin(k_f + q/2) \sin q/2$$



$$\Rightarrow E_{up} = \begin{cases} 2J \sin(k_f + q/2) \sin q/2 & \text{for } q < \pi - 2k_f \\ 2J \sin q/2 & \text{for } q > \pi - 2k_f . \end{cases}$$

③ for spin change $S_\pm$, $M \to M \pm 1$

$E_G(M \mp 1) \pm h - E_G(M) = 0$. So when $-hS_z$ is counted in the Hamiltonian, the ground state energy shift goes to zero.

For the AFM, we need to shift the above p-h continuum by π