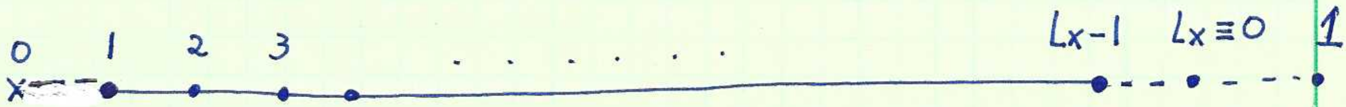
Lect 9: Bulk & Edge correspondence

Solve Harper Equation with an open boundary condition

$$H = -t_x \sum_{m,n} C^{+}_{m+1,n} C_{m,n} - t_y e^{i2\pi \Phi/L_y} \sum_{m,n} C^{+}_{m,n+1} C_{m,n} e^{i2\pi\phi m}$$

$+ h.c.$ $\longrightarrow$ x-direction



open boundary: wavefunction vanishes at $m = 0$, and L_x. L_x is equivalent to 0.

(m: site index (x-component))

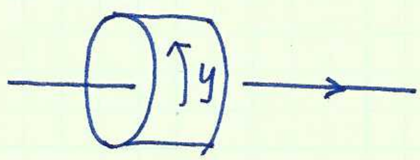
define $C_{m,n} = \frac{1}{\sqrt{L_y}} \sum_{k_y} e^{ik_y n} C_m(k_y)$

and $C_m(k_y) = \frac{1}{\sqrt{L_y}} \sum_{n} e^{-ik_y n} C_{m,n}$

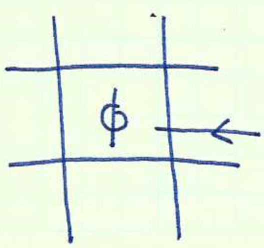
For states $|\psi\rangle_{k_y} = \sum_m \psi_m(k_y, \Phi)\, C^{+}_m(k_y)|0\rangle \rightarrow$ Schröding Eq

$$(*)\quad \boxed{-t_x\{\psi_{m+1}(k_y,\Phi) + \psi_{m-1}(k_y,\Phi)\} - 2t_y \cos(k_y - 2\pi\frac{\Phi}{L_y} - 2\pi\phi m)\,\psi_m(k_y) = E\,\psi_m(k_y)}$$

Φ: A flux to control twisted boundary condition



$\longrightarrow$ Φ (twisted boundary condition)



$\leftarrow$ magnetic flux per plaquett

$\phi = P/q$

Change (*) to the form of Harper Eq.

$$\begin{pmatrix} \psi_{m+1} \\ \psi_m \end{pmatrix} = M_m(\epsilon) \begin{pmatrix} \psi_m \\ \psi_{m-1} \end{pmatrix} \quad \text{with } M_m(\epsilon) = \begin{bmatrix} -\epsilon - 2r\cos\left(k_y - 2\pi\phi m - 2\pi \Phi/L_y\right), & - \\ 1, & 0 \end{bmatrix}$$

Set the length along x-direction L_x integer time of q, i.e. $L_x = ql \Rightarrow$

$$\begin{pmatrix} \psi_2 \\ \psi_1 \end{pmatrix} = M_1(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \quad \begin{pmatrix} \psi_3 \\ \psi_2 \end{pmatrix} = M_2(\epsilon) M_1(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} \cdots$$

$$\Rightarrow \begin{pmatrix} \psi_{q+1} \\ \psi_q \end{pmatrix} = M_q(\epsilon) M_{q-1}(\epsilon) \cdots M_1(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = M(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

define $M(\epsilon) = M_q(\epsilon) M_{q-1}(\epsilon) \cdots M_1(\epsilon)$,

$$\Rightarrow \begin{pmatrix} \psi_{ql+1} \\ \psi_{ql} \end{pmatrix} = \begin{pmatrix} \psi_{L_x+1} \\ \psi_{L_x} = 0 \end{pmatrix} = [M(\epsilon)]^l \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = [M(\epsilon)]^l \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}$$

open boundary $\psi(L_x) = \psi(0) = 0$.

$$\Rightarrow \begin{pmatrix} \psi_{L_x+1} \\ 0 \end{pmatrix} = \begin{pmatrix} M^l_{11} & M^l_{12} \\ M^l_{21} & M^l_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix} \Rightarrow \boxed{M^l_{21}(\epsilon) = 0}$$

determine eigen-energy

$$M^l = [M_q(\epsilon) \cdots M_1(\epsilon)]^l, \quad \text{since } M \sim \begin{bmatrix} \epsilon + \text{const} & -1 \\ 1 & 0 \end{bmatrix}$$

$$M^l \sim \begin{bmatrix} \epsilon^{ql}, & \epsilon^{ql-1} \\ \epsilon^{ql-1}, & \epsilon^{ql-2} \end{bmatrix} \sim \begin{bmatrix} \epsilon^{L_x}, & \epsilon^{L_x-1} \\ \epsilon^{L_x-1}, & \epsilon^{L_x-2} \end{bmatrix}$$

$M^{\ell}_{21}(\epsilon)$ is a polynomial of ϵ with power of $L_x - 1$, thus it has $L_x - 1$ roots. These roots are real since they are eigenvalues of 1D lattic Hamiltonian, which include both edge and bulk states. we still need to figure out the criterion for edge states.

$\boxed{\text{☆ HW: Prove that the condition for the edge state energy is that the roots of } M_{21}(\epsilon) = 0. \text{ For these } \epsilon\text{'s, } \psi_q(\epsilon) = \psi_{2q}(\epsilon) = \cdots = \psi_{q\ell}(\epsilon) = 0.}$

Proof: ① A root of $M_{21}(\epsilon) = 0$ is also a root of $[M^{\ell}]_{21}(\epsilon) = 0$. Thus the roots of $M_{21}(\epsilon)$ correspond to eigenenergies. This is because

$$\begin{bmatrix} M_{11} & M_{21} \\ 0 & M_{22} \end{bmatrix} \begin{bmatrix} M_{11} & M_{21} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11}^2 & \# \\ 0 & M_{22}^2 \end{bmatrix} \Rightarrow \begin{bmatrix} M_{11} & M_{21} \\ 0 & M_{22} \end{bmatrix}^{\ell} = \begin{bmatrix} M_{11}^{\ell}, & \# \\ 0, & M_{22}^{\ell} \end{bmatrix}$$

$M(\epsilon) = M_q(\epsilon) \cdots M_1(\epsilon) \Rightarrow M_{21}(\epsilon)$ is a $q-1$ order polynomial

$\Rightarrow$ there're $q-1$ roots, denoted as $\mu_j : j = 1, 2, \cdots q-1$.

For these roots $\begin{bmatrix} \psi_{q\ell+1} \\ 0 \end{bmatrix} = \begin{bmatrix} M_{11}^{\ell}(\mu_j) & \# \\ 0, & M_{22}^{\ell}(\mu_j) \end{bmatrix} \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \Rightarrow \dfrac{\psi_{q\ell+1}}{\psi_1} = M_{11}^{\ell}(\mu_j)$

① if $|M_{11}(\mu_j)| < 1 \Rightarrow \mu_j$ belongs to a localized state at left edge

② if $|M_{11}(\mu_j)| > 1 \rightarrow \mu_j$ belongs to a state localized at right edge

③ if $|M_{11}(\mu_j)| = 1 \rightarrow$ degenerate localized edge / bulk state. Merging point!

Next we prove that those ϵ satisfying $M_{21}(\epsilon) = 0$ does not live in the continuum of energy bands. We need to figure out the band edges. Now let's switch to the $\boxed{\text{periodical boundary condition}}$.

According to Bloch theory $\Rightarrow \begin{cases} \psi_{m+q}(\epsilon) = \rho(\epsilon)\,\psi_m(\epsilon) \text{ with } \rho(\epsilon) = 1 \\ \text{periodicity } q \end{cases}$

$$\begin{pmatrix} \psi_{q+1} \\ \psi_q \end{pmatrix} = M(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = \rho(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} \Rightarrow \rho(\epsilon) \text{ is an eigenvalue of } M(\epsilon).$$

$$\det \begin{pmatrix} M_{11}(\epsilon) - \rho, & M_{12} \\ M_{21}, & M_{22}(\epsilon) - \rho \end{pmatrix} = 0 \Rightarrow \rho^2 - tr(M)\,\rho + \det M = 0$$

$$\det M = \det(M_q) \cdots \det M_1 = 1 \Rightarrow \boxed{\rho^2(\epsilon) - (M_{11}(\epsilon) + M_{22}(\epsilon))\,\rho + 1 = 0}$$

The requirement of $|\rho_{(\epsilon)}| = 1$ for bulk states $\Rightarrow \boxed{(tr\, M)^2 \le 4,}$ (since $tr\, M$ is real).

for bulk state energy ϵ.

For the energy "μ_j" satisfying $M_{21}(\mu_j) = 0$. Since $\det M = 1 \Rightarrow$

$$M_{11}(\mu_j)\; M_{22}(\mu_j) = 1$$

thus for $\mu_j \Rightarrow \left[M_{11}(\mu_j) + M_{22}(\mu_j) \right]^2 = (M_{11} + \frac{1}{M_{11}})^2 \ge 4$

$\Rightarrow$ These energies "μ_j" do not belong to the bulk state energies "ϵ" for Bloch states. Thus μ_j lies in the band gaps.

Winding # on the Riemann surface

1 Bloch wavefunction

$$\begin{pmatrix} \psi_{q+1}(\epsilon) \\ \psi_q(\epsilon) \end{pmatrix} = M(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix} = \rho(\epsilon) \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}$$

due to the Bloch theory $|\rho(\epsilon)| = 1$.

$\rho(\epsilon)$ is an eigenvalue of $M(\epsilon)$.

Analytic continuation on $\epsilon \to$ complex z

define $\Delta(z) = \mathrm{tr}\, M(z) \Rightarrow$

$$\boxed{\rho^2(z) - \Delta(z)\rho(z) + 1 = 0 \Rightarrow \rho(z) = \frac{1}{2}\left[\Delta(z) - \sqrt{\Delta^2(z) - 4}\right]}$$

we have not specify Rieman surface yet.

1 When we restrict z to be real "ϵ", the physical states (Bloch waves) correspond to $\Delta^2(\epsilon) - 4 \leq 0$. Since there are q-bands, we should have $2q$ zeros, and there are real. ($\Delta(\epsilon)$ is ϵ's q power polynomial)

2 Now we extend ϵ to complex z, and define

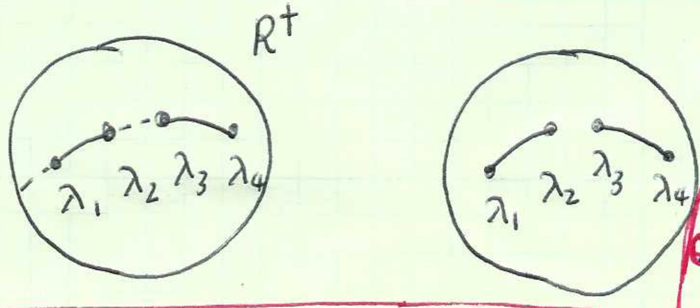
$$\omega = \Delta^2(z) - 4 = (z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2q})$$

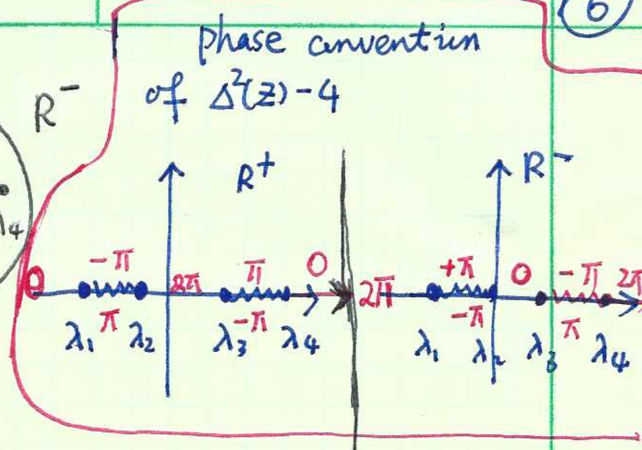
A. if $\omega < 0$, then $|\rho(z)| = 1$, corresponding to $e^{\pm ikx}$

$\rightarrow$ Bloch wave bulk states

B: Branch-cut and Rieman surfaces

branch convention $\sqrt{\Delta^2(z) - 4}$ $\begin{cases} >0 \\ <0 \end{cases}$ as $z \to -\infty$ in $\begin{cases} R^+ \\ R^- \end{cases}$





$$\boxed{\text{The gap region:} \quad \lambda_{2j} < z < \lambda_{2j+1}}$$

Sign convention in R^+ :

$\sqrt{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2q})}$ ① if $z<\lambda_1$, $\sqrt{\Delta^2(z)-4}$ is defined as >0,

then in the gap $\lambda_2 < z < \lambda_3$, $\sqrt{\Delta^2(z)-4} < 0$, ..., and so on.

$\Rightarrow$ for z real in R^+, and lies in the j-th gap $(\lambda_{2j} < z < \lambda_{2j+1})$

$$\left.\begin{array}{ll} \Rightarrow & (-)^j \sqrt{\Delta^2(z)-4} > 0 \quad \text{for } R^+ \\ \text{similarly} & (-)^j \sqrt{\Delta^2(z)-4} < 0 \quad \text{for } R^- \end{array}\right\}$$

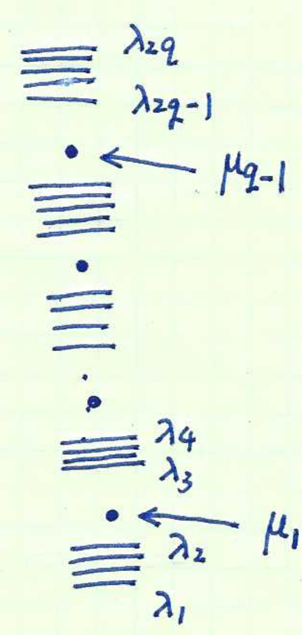
The bulkspectra consist of q-bands, we denote them as

$[\lambda_1, \lambda_2]$, $[\lambda_3\ \lambda_4]$, $\cdots$ $[\lambda_{2q-1}, \lambda_{2q}]$

$\mu_j \in [\lambda_{2j}, \lambda_{2j-1}] \leftarrow j$th gap

$\uparrow$

edge state energy



At the edge state energy μ_j,

$$\boxed{\begin{aligned} \sqrt{\Delta^2(\mu_j)-4} &= \pm(-)^j \, |M_{11}(\mu_j) - M_{22}(\mu_j)| \\ = \sqrt{(M_{11}+M_{22})^2-4} & \qquad (\pm \text{ for } R^\pm) \end{aligned}}$$

Based on $\begin{pmatrix}\psi_{q+1}\\ \psi_q\end{pmatrix} = \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix} = \rho(\epsilon)\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix}$

Normalization convention: set $\psi_1 = 1 \Rightarrow \psi_q = M_{21} + M_{22}\psi_0 = \rho(\epsilon)\psi_0$

$\Rightarrow \psi_0 = \dfrac{M_{21}}{\rho(\epsilon) - M_{22}}$ and $\psi_q(z) = M_{21}\psi_1 + M_{22}\psi_0 = M_{21}\left[1 + \dfrac{M_{22}}{\rho(z) - M_{22}}\right]$

$$\Rightarrow \boxed{\psi_q(z) = M_{21}\frac{\rho(z)}{\rho(z) - M_{22}} = M_{21}(z)\frac{M_{11}(z) + M_{22}(z) - \sqrt{\Delta^2(z) - 4}}{M_{11}(z) - M_{22}(z) - \sqrt{\Delta^2(z) - 4}}}$$

In order to have $\psi_q(\mu_j) = 0$, we need

$$\begin{cases} M_{21}(\mu_j) = 0 & ① \\ M_{11}(\mu_j) - M_{22}(\mu_j) - \sqrt{\Delta^2(\mu_j) - 4} \neq 0 & ② \end{cases}$$

the $\sqrt{\cdots}$ in Condition (2) needs to be determined by μ_j in R^+ or R^-.

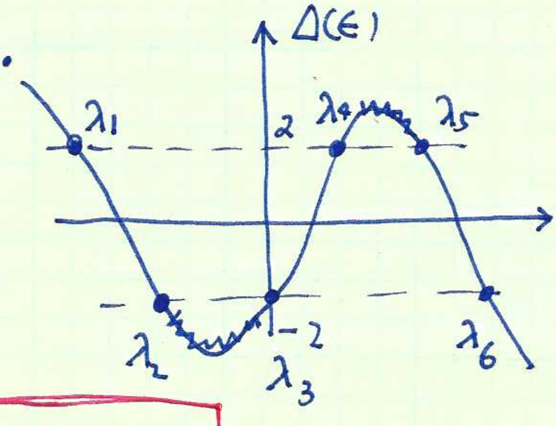
⊛ Now let us determine the sign of $\Delta(\epsilon)$ for ϵ inside the gap $[\lambda_{2j}, \lambda_{2j+1}]$.

Set $\epsilon \to -\infty$, then $M \to \begin{bmatrix}-\epsilon & 0\\ 0 & 0\end{bmatrix}^q \Rightarrow \Delta(\epsilon) \sim (-\epsilon)^q \to +\infty$.

By continuity, $\Delta(\epsilon)$ intersects the line $y = 2$ from above at λ_{2j}. ~~By~~ it goes down to cross $y = -2$ at $y = -2$.

Then $\Delta(\epsilon) < -2$ at ϵ in $[\lambda_2, \lambda_3]$.

Similarly, we have $\Delta(\epsilon) > 2$ for ϵ in $[\lambda_4, \lambda_5]$, $\cdots$ and so on.



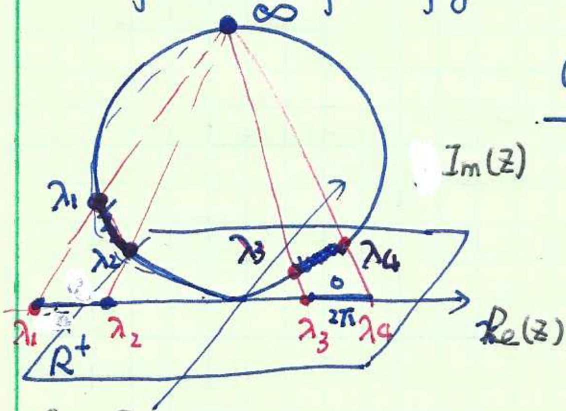
$$\Rightarrow \boxed{\Delta(\epsilon) \le -2 \text{ for } j \text{ odd}, \quad \ge 2 \text{ for } j \text{ even}.}$$

	j odd	j even
R^+	$\left.\begin{array}{l}\Delta(\mu_j) < -2 \\ -\sqrt{\Delta^2-4} > 0\end{array}\right\} \Rightarrow \lvert\rho\rvert<1$	$\left.\begin{array}{l}\Delta(\mu_j) > 2 \\ -\sqrt{\Delta^2-4} < 0\end{array}\right\} \Rightarrow \lvert\rho\rvert<1$
R^-	$\left.\begin{array}{l}\Delta(\mu_j) < -2 \\ -\sqrt{\Delta^2-4} < 0\end{array}\right\} \Rightarrow \lvert\rho\rvert>1$	$\left.\begin{array}{l}\Delta(\mu_j) > 2 \\ -\sqrt{\Delta^2-4} > 0\end{array}\right\} \Rightarrow \lvert\rho\rvert>1$

$\Rightarrow$ ① if $\mu_j \in R^+$, $|\rho|<1$, the edge state is localized around the left edge $x \simeq 1$.

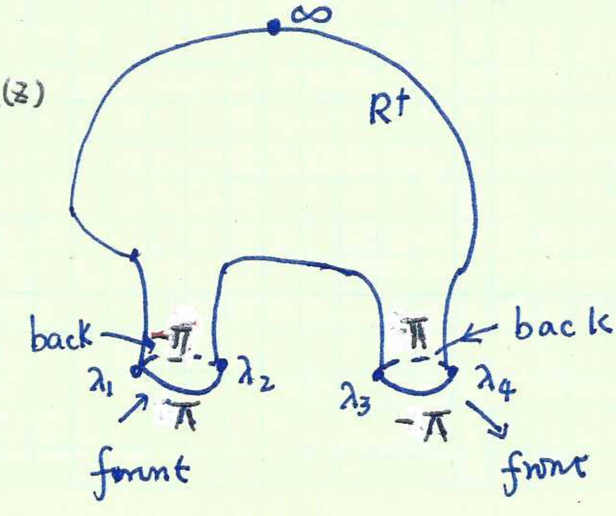
② if $\mu_j \in R^-$, $|\rho|>1$, the edge state is localized around the right edge $x \simeq L_x - 1$.

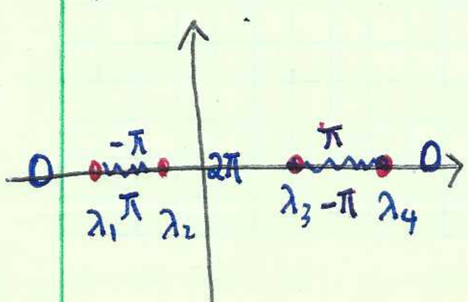
Let's first ampactify the Riemann surface



$\xrightarrow{\text{Cut the ballon along the line segments}}$ $\underbrace{\lambda_1 \lambda_2}$ and $\underbrace{\lambda_3 \lambda_4}$

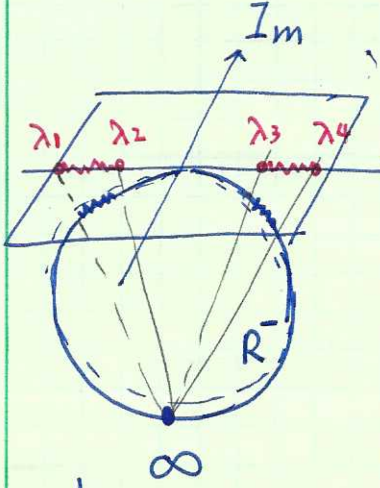
phase of $\Delta^2(z)-4$ $\rightarrow$

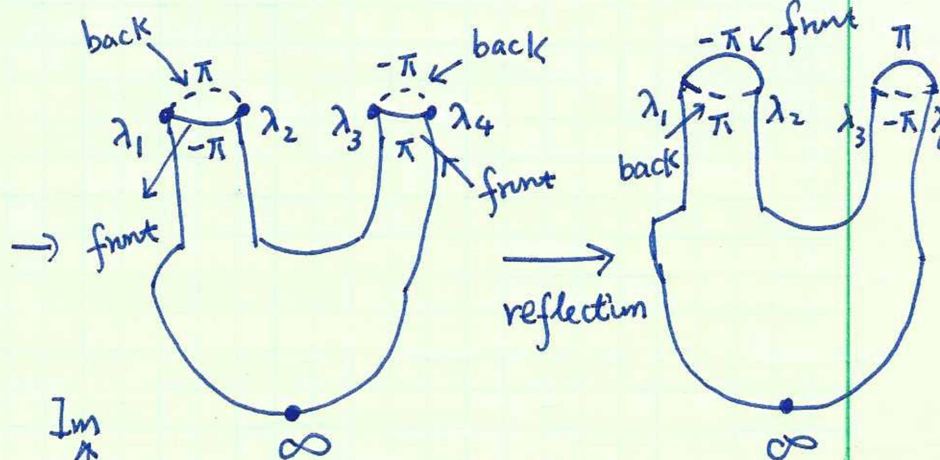




Similarly for R^-

cut the ballon along $\lambda_1\lambda_2$ and $\lambda_3\,\lambda_4$





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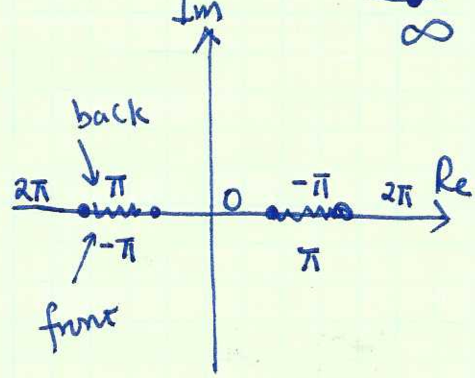
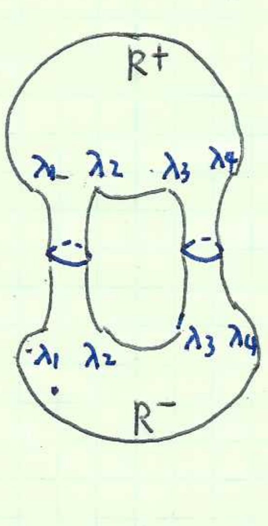
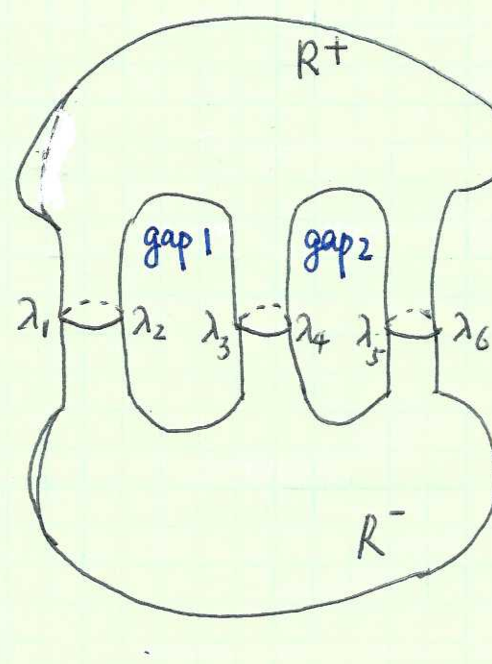
phase of $\Delta^2(z)-4$



glue them to form a torus



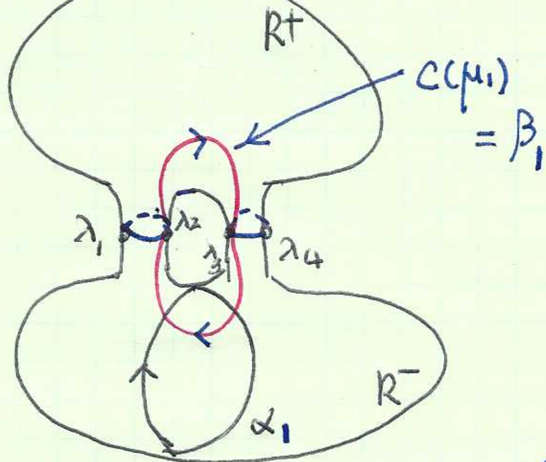
more gaps

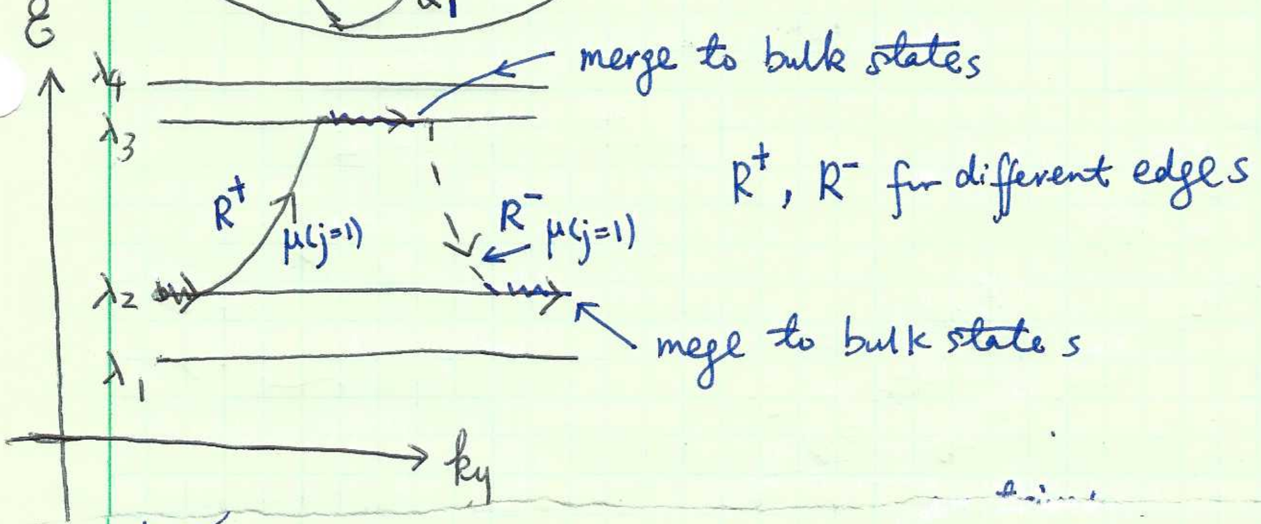


The above torus construction is for a fixed value of k_y. If k_y varys, the vaules of $\lambda_j(k_y)$ also varies. As long as gap does close (q: odd), the topology doesn't change.

* Consider let k_y to vary, the edge state energy solutions μ_j also varies. (μ_j are zeros: $\psi_q(\mu_j) = 0$). As k_y varies from $0 \to 2\pi$, μ_j should form a closed up. denoted as $C(\mu_j)$.

$C(\mu_1)$ moves from R_+ to R_- and from R_- to R_+

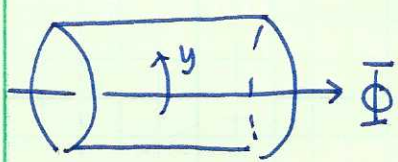




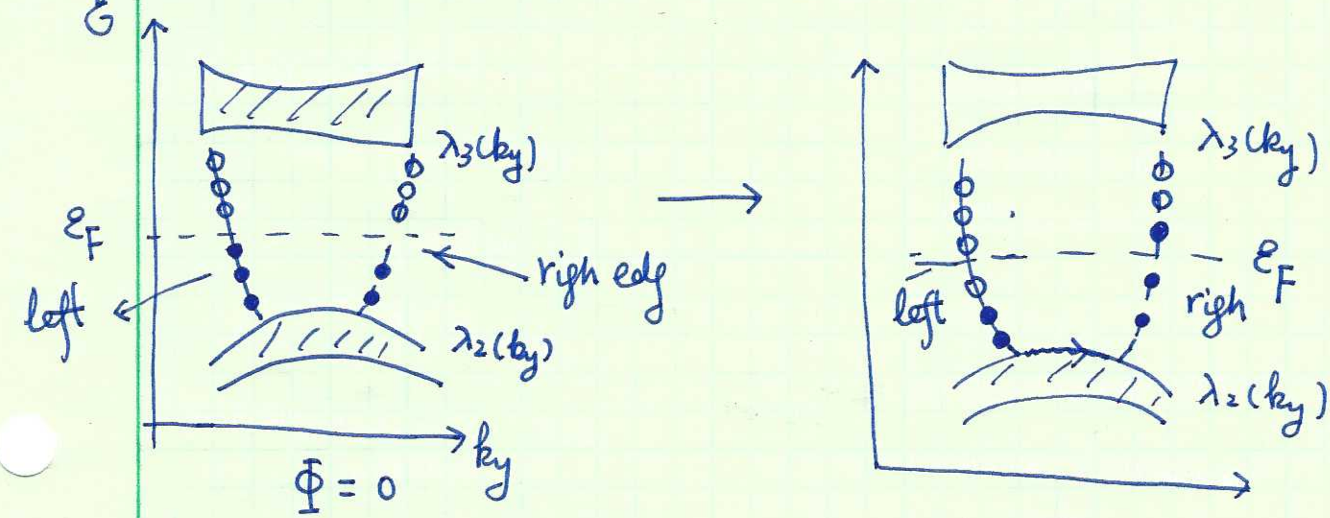
Defining winding #: For torus (for simplicity, consider genus = 1), there're two fundemental (non-trivial) loops. α_1 and β_1, as shown above. As k_y varies from $0 \to 2\pi$, $C(\mu_1)$ must go arond β integer times, i.e. $C(\mu_1) \simeq \beta_1^t$. We can define the winding #

$$I[\alpha_1, C(\mu_1)] = t.$$

According to Laughlin gauge argument, σ_{xy} corresponds to the charge pump from one edge to the other as Φ varying from $0 \to 1$ (hc/e)



This equavilent $k_y \to k_y - 2\pi \Phi/L_y$, thus k_y only changes on step.



$$I_y = \frac{e}{h}\frac{\Delta E}{\Delta \Phi} = \sigma_{yx} V_x$$

dimestionless $\Delta\Phi$ in terms of (hc/e)

$\Delta E = n e V_x$

$\Rightarrow \sigma_{yx} = n\, e^2/h$

$\Rightarrow$ ~~we need~~ a nontrivial loop β_1 contribute a e^2/h.

$$\Rightarrow \boxed{\sigma_{xy} = \frac{e^2}{h} I[\alpha_1, C(\mu_1)].}$$

The link (winding) number is actually can be positive or negative say. We fix the orientation of α_1, — the two different orientations of the $C[\mu_1]$, i.e. β_1 or β_1^{-1}, give $\sigma_{xy} = \pm e^2/h$.

Furthure proof: Winth in each band $[\lambda_{2j+1}, \lambda_{2j}]$, $\Delta(\epsilon)$ has one and only one zero. Since $\rho^2(\epsilon) - \Delta(\epsilon)\rho(\epsilon) + 1 = 0$, the zero $\overset{\text{for}}{\frown}$ of $\Delta(\epsilon_0) = 0$, we have $\rho^2(\epsilon_0) = -1$, or $\boxed{\rho(\epsilon_0) = \pm i.}$

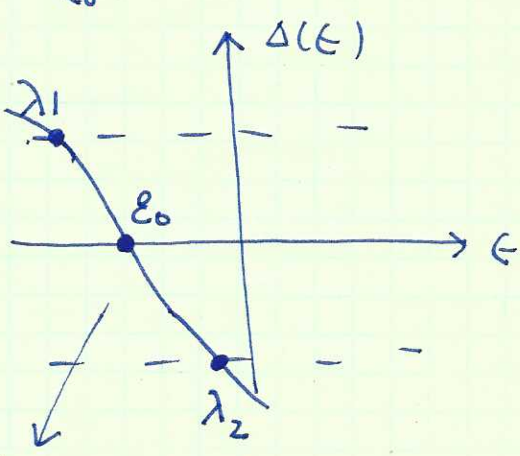
$\rho(\epsilon_0)$ is the Bloch wave phase factor, i.e $\begin{pmatrix} \psi_{q+1}(\epsilon) \\ \psi_q(\epsilon) \end{pmatrix} = \pm i \begin{pmatrix} \psi_1(\epsilon) \\ \psi_0(\epsilon) \end{pmatrix}$.

Acording to the Harper Eq

$$-t_x\{\psi_{m+1}(k_y, \Phi) + \psi_{m-1}(k_y, \Phi)\} - 2t_y\cos\left(k_y - 2\pi\frac{\Phi}{L_y} - 2\pi\phi m\right)\psi_m(k_y, \Phi)$$
$$= \epsilon\,\psi_m(k_y)$$

This Eq is real, and have the periodicity of q, $\therefore$ ϵ is determined by ρ, ie $\epsilon(\rho)$. Thus $\epsilon_0 = \epsilon(\rho = \pm i)$. Since the Harper Eq is real, $\Rightarrow \underbrace{\epsilon(\rho = +i) = \epsilon(\rho = -i)}_{\epsilon_0 =}$, thus there's only one zero ϵ_0.

$\Rightarrow$



only one zero

$\rho = e^{i k_x q}$

Check the Hamiltonian on Page 6 of lecture 8

$H(k_x, k_y) = H^*(-k_x, k_y)$

$\Rightarrow$ theire eigenvalues are the same, for ρ and ρ^* i.e $e^{\pm i k_x q}$.