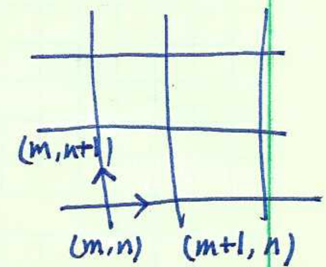
Lect 8: Hofstadter problem

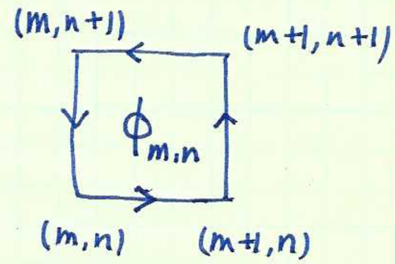
Consider magnetic field in a square lattice



$$H = K_x + K_y + h.c$$

$$K_x = \sum_{m,n} C^\dagger_{m+1,n} C_{m,n} e^{i\theta^x_{m,n}}, \quad \text{where } \theta^x_{m,n} = \frac{e}{\hbar c}\int_m^{m+1} A_x dx$$

$$K_y = \sum_{m,n} C^\dagger_{m,n+1} C_{m,n} e^{i\theta^y_{m,n}} \qquad \theta^y_{m,n} = \frac{e}{\hbar c}\int_n^{n+1} A_y dy$$



$$\Rightarrow \theta^x_{m,n} + \theta^y_{m+1,n} - \theta^x_{m,n+1} - \theta^y_{m,n}$$

$$= \frac{e}{\hbar c}\oint \vec{A}\cdot d\vec{l} = 2\pi \frac{\phi_{m,n}}{hc/e} \leftarrow \text{flux}$$

For simplicity, we drop the unit hc/e, and only write down ϕ.

K_x and K_y do not commute: for a state $|mn\rangle = C^\dagger_{mn}|0\rangle$

$$K_x K_y |mn\rangle = K_x e^{i\theta^y_{m,n}} |m,n+1\rangle = e^{i\theta^x_{m,n+1} + i\theta^y_{m,n}} |m+1, n+1\rangle$$

$$K_y K_x |mn\rangle = K_y e^{i\theta^x_{m,n}} |m+1,n\rangle = e^{i\theta^x_{m,n} + i\theta^y_{m+1,n}} |m+1,n+1\rangle$$

$$\boxed{\underbrace{K_y K_x = K_x K_y\, e^{i(\theta^x_{m,n} + \theta^y_{m+1,n} - \theta^x_{m,n+1} - \theta^y_{m,n})}}_{|mn\rangle} \underbrace{=}_{|mn\rangle} \underbrace{e^{i2\pi\phi_{m,n}}}_{|mn\rangle}}$$

if for uniform flux $\phi_{m,n} = \phi = p/q \Rightarrow$

$$\boxed{K_y K_x = K_x K_y \cdot e^{i2\pi\phi}}$$

Lattice version of the magnetic translation operator

define $\hat{T}_x = \sum_{mn} C^{\dagger}_{m+1,n} C_{m,n} e^{i\chi^x_{m,n}}$

$\hat{T}_y = \sum_{mn} C^{\dagger}_{m,n+1} C_{mn} e^{i\chi^y_{m,n}}$

$[K_x, \hat{T}_x]=0$

$\Downarrow$ $K_x \hat{T}_x |m\, n\rangle = K_x e^{i\chi^x_{m,n}} |m+1, n\rangle = e^{i(\theta^x_{mn} + \chi^x_{m+1,n})} |m+2, n\rangle$

$\hat{T}_x K_x |m,n\rangle = e^{i(\chi^x_{m,n} + \theta^x_{m+1,n})} |m+2, n\rangle$

$\Rightarrow \quad \theta^x_{m+1,n} - \theta^x_{mn} = \chi^x_{m+1,n} - \chi^x_{m,n} \Rightarrow \boxed{\Delta_x \chi^x_{mn} = \Delta_x \theta^x_{m,n}}$

$[K_y, \hat{T}_y]=0$

$\Downarrow$ similarly $\boxed{\Delta_y \chi^y_{m,n} = \Delta_y \theta^y_{m,n}}$

$[K_y, \hat{T}_x]=0$ $\quad K_y \hat{T}_x |m\, n\rangle = K_y e^{i\chi^x_{m,n}} |m+1, n\rangle = e^{i\theta^y_{m+1,n} + i\chi^x_{m,n}} |m+1, n+1\rangle$

$\hat{T}_x K_y |m\, n\rangle = \hat{T}_x e^{i\theta^y_{m,n}} |m, n+1\rangle = e^{i\chi^x_{m,n+1} + i\theta^y_{m,n}} |m+1, n+1\rangle$

$\Rightarrow \quad \theta^y_{m+1,n} - \theta^y_{m,n} = \chi^x_{m,n+1} - \chi^x_{m,n} \Rightarrow \boxed{\Delta_y \chi^x_{mn} = \Delta_x \theta^y_{mn} = \Delta_y \theta^x_{mn} + 2\pi \phi_{mn}}$

Similarly $[K_x, \hat{T}_y]=0 \Rightarrow \boxed{\Delta_x \chi^y_{m,n} = \Delta_y \theta^x_{mn} = \Delta_x \theta^y_{m,n} - 2\pi \phi_{mn}}$

we solve these constrains as

$$\boxed{\chi^x_{mn} = \theta^x_{mn} + 2\pi n \phi_{m,n}, \qquad \chi^y_{mn} = \theta^y_{m,n} - 2\pi m \phi_{m,n}}$$

Again $\hat{T}_x, \hat{T}_y$ don't commute,

$$\hat{T}_x \hat{T}_y |mn\rangle = T_x e^{i\chi^y_{m,n}} |m,n+1\rangle = e^{i\chi^x_{m,n+1} + \chi^y_{m,n}} |m+1,n+1\rangle$$

$$\hat{T}_y \hat{T}_x |mn\rangle = T_y e^{i\chi^x_{m,n}} |m+1,n\rangle = e^{i\chi^x_{m,n} + i\chi^y_{m+1,n}} |m+1,n+1\rangle$$

$$\Rightarrow \hat{T}_x \hat{T}_y |mn\rangle = e^{-i(\chi^x_{m,n} + \chi^y_{m+1,n} - \chi^x_{m,n+1} - \chi^y_{m,n})} \hat{T}_y \hat{T}_x |mn\rangle$$

$$\chi^x_{m,n} + \chi^y_{m+1,n} - \chi^x_{m,n+1} - \chi^y_{m,n} = \theta^x_{m,n} + 2\pi n \phi_{m,n} - \theta^x_{m,n+1} - 2\pi(n+1)\phi_{m,n+1}$$

$$+ \theta^y_{m+1,n} - 2\pi(m+1)\phi_{m+1,n} - \theta^y_{m,n} + 2\pi m \phi_{m,n}$$

if $\phi_{m,n}$ is a constant $\Rightarrow$ $\chi^x_{m,n} + \chi^y_{m+1,n} - \chi^x_{m,n+1} - \chi^y_{m,n}$

$$= (\theta^x_{m,n} - \theta^x_{m,n+1} + \theta^y_{m+1,n} - \theta^y_{m,n}) - 4\pi\phi$$

$$= -2\pi\phi$$

$$\Rightarrow \boxed{\hat{T}_x \hat{T}_y = \hat{T}_y \hat{T}_x e^{i2\pi\phi}}$$

Consider the Landau gauge $A_x = 0$, $A_y = Bx = 2\pi\phi m$

$$\Rightarrow \boxed{\begin{cases} \theta^x_{m,n} = 0, \quad \theta^y_{m,n} = 2\pi m \phi \\ \chi^x_{m,n} = 2\pi n \phi, \quad \chi^y_{m,n} = 0 \end{cases}}$$

$$\Rightarrow \boxed{\begin{aligned} \hat{T}_x &= \sum_{m,n} C^\dagger_{m+1,n} C_{m,n} e^{i2\pi n\phi} \\ \hat{T}_y &= \sum_{m,n} C^\dagger_{m,n+1} C_{m,n} \end{aligned}}$$

Then $[T_x^q, T_y] = 0$. We define magnetic BZ $\begin{cases} 0 \le k_x \le 2\pi/q \\ 0 \le k_y \le 2\pi \end{cases}$

$$H|\vec{k}\rangle = E(\vec{k})|\vec{k}\rangle$$

with $\hat{T}_x^q |\vec{k}\rangle = e^{-ik_x q a}|\vec{k}\rangle$, $\quad \hat{T}_y|\vec{k}\rangle = e^{-ik_y a}|\vec{k}\rangle$

HW:
★ on the lattice with each plaquette flux $\phi = p/q$, for each state $|\vec{k}\rangle = |k_x, k_y\rangle$, there exists at least q-fold degeneracy with different k_y but the same k_x.

Proof: $\quad T_y(T_x|\vec{k}\rangle) \overset{e^{i2\pi\phi}}{=} T_x(T_y|\vec{k}\rangle) = e^{i(k_y - 2\pi\phi)}\hat{T}_x|\vec{k}\rangle$

$\Rightarrow T_x|\vec{k}\rangle \sim |k_x, k_y - 2\pi\phi\rangle$.

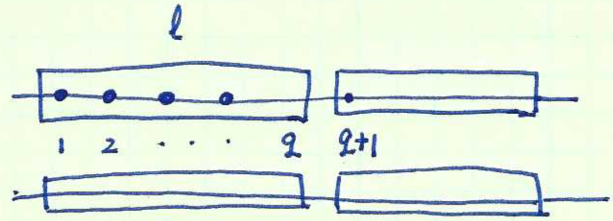
Clearly $T_x|\vec{k}\rangle$, $T_x^2|\vec{k}\rangle, \cdots \quad T_x^{q-1}|\vec{k}\rangle$ are degenerate since $[T_x, H] = 0$. Their lattice momenta are $|k_x, k_y - 2\pi\phi\rangle, \cdots$

$|k_x, k_y - 2\pi(q-1)\phi\rangle$.

Harper Equation

$$H = -t_x \sum_{m,n} c^{\dagger}_{m+1,n} c_{m,n} - t_y \sum_{m,n} c^{\dagger}_{m,n+1} c_{m,n} e^{i2\pi m \phi}$$

Fourier transform:



$$m = l q + r$$

l: index of unit-cell, r: index of site insider the unit cell

$$c_{lq+r,n} = \frac{1}{\sqrt{L_x L_y / q}} \sum_{k_x,k_y} e^{i k_x l q + i k_y n} \, c_r(k_x,k_y)$$

$$\Rightarrow \quad H = -\sum_r \sum_{k_x,k_y} \left[t_{x,r} \, c^{\dagger}_{r+1}(k_x,k_y) \, c_r(k_x,k_y) + t^{*}_{x,r} \, c^{\dagger}_r(k_x,k_y) \, c_{r+1}(k_x,k_y) \right]$$

$$- \sum_{r,k_x,k_y} 2 t_y \cos(k_y - 2\pi r \phi) \, c^{\dagger}_r(k_x,k_y) \, c_r(k_x,k_y)$$

where
$$\begin{cases} t_{x,r} = t_x & \text{for } r = 1, 2, \cdots q-1 \\ t_{x,r=q} = t_x e^{-i k_x q} & \end{cases}$$

$$H = \sum_{k_x,k_y} \left[c^{\dagger}_1(k_x,k_y) \cdots c^{\dagger}_r(k_x,k_y) \right] H(k_x,k_y) \begin{bmatrix} c_1(k_x,k_y) \\ \vdots \\ \vdots \\ c_r(k_x,k_y) \end{bmatrix}$$

with

$$H(k_x,k_y)=\begin{bmatrix} \boxed{-2t_y\cos(k_y-2\pi\phi), \quad -t_x} & ; & 0 & , & & -t_x e^{-ik_xq} \\ \boxed{-t_x \quad , -2t_y\cos(k_y-4\pi\phi), \quad -t_x} & & , \cdots & 0 & & 0 \\ & & & & & \vdots \\ -t_x e^{ik_xq} \quad , & 0 & , & 0 & , \cdots & \boxed{-t_x \, , -2t_y\cos k_y} \end{bmatrix}$$

The eigen wavefunction $\psi^{(t)}(k_x,k_y)=\begin{bmatrix}\psi_1^{(t)}\\ \vdots \\ \psi_r^{(t)}\end{bmatrix}$

$t=1,2,\cdots r$

or the corresponding state $|\psi^{(t)}(k_x,k_y)\rangle = \sum_{r=1}^{q} \psi_r^{(t)}(k_x,k_y)\, C_r^{\dagger}(k_x,k_y)\,|0\rangle$.

In this Rep, it's explicit that $H(k_x+2\pi/q, k_y) = H(k_x,k_y)$.

But the relation of $H(k_x,k_y)$ with $H(k_x, k_y+2\pi/q)$ is not explicit. We can use a different Rep

$$C_{lq+r,n} = \frac{1}{\sqrt{L_xL_y/q}} \sum_{k_x k_y} e^{ik_xm+ik_yn}\, C_r'(k_x,k_y)$$

$$\Rightarrow C_r'(k_x,k_y) = e^{-ik_xr}\, C_r(k_x,k_y), \text{ then}$$

$$H=\sum_{k_x,k_y}[C_1'^{\dagger}(k_x,k_y)\cdots C_r'^{\dagger}(k_x,k_y)]\, H'(k_x,k_y) \begin{bmatrix} C_1'(k_x,k_y)\\ \vdots \\ C_r'(k_x,k_y)\end{bmatrix}$$

with

$$H'(k_x, k_y) = \begin{bmatrix} -2t_y\cos(k_y - 2\pi\phi), & -t_x e^{+ik_x}, & \cdots & -t_x e^{-ik_x} \\ -t_x e^{-ik_x}, & -2t_y\cos(k_y - 4\pi\phi), & -t_x e^{ik_x}, & \cdots \\ & & & \\ -t_x e^{ik_x}, & \cdots, & \cdots & -t_x e^{-ik_x}, \ -2t_y \cos k_y \end{bmatrix}$$

it's clear $\quad H'(k_x, k_y) = T^{-1} H'(k_x, k_y - 2\pi\phi)\, T$

where $T^{-1} = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \\ & & 1 & 0 \end{bmatrix} \rightarrow \begin{array}{l} T_{ij} = \delta_{i+1,j} \\ (T^{-1})_{ij} = \delta_{i-1,j} \end{array}$

$T = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} H'_{ij}(k_x, k_y) = \delta_{i-1,j'} H'_{j'l'}(k_x, k_y - 2\pi\phi)\, \delta_{l'+1,j} \\ = H'_{i-1,j-1}(k_x, k_y - 2\pi\phi) \end{array}$

$$\Rightarrow \boxed{H'_{(k_x,k_y)} \psi^{(t)'}(k_x, k_y) = E^{(t)}(k_x, k_y)\, \psi^{(t)'}(k_x, k_y)}$$

$$\Rightarrow H'(k_x, k_y - 2\pi\phi)\left[T \psi^{(t)'}(k_x, k_y)\right] = E^{(t)}(k_x, k_y)\left(T \psi^{(t)'}(k_x, k_y)\right)$$

$$\psi'^{(t)}(k_x, k_y - 2\pi\phi) = T \psi'^{(t)}(k_x, k_y) = \begin{bmatrix} \psi_2'^{(t)} \\ \psi_3'^{(t)} \\ \vdots \\ \psi_1'^{(t)} \end{bmatrix} (k_x, k_y)$$

In the Rep of H, we have $H(k_x, k_y) = H(k_x + \frac{2\pi}{q}, k_y)$

and in the rep of H', we have $\boxed{H'(k_x, k_y) = T^{-1} H'(k_x, k_y - 2\pi\phi) T}$

with $T_{ij} = \delta_{i+1,j}$, $\phi = p/q$.

since $C'_r(k_x, k_y) = e^{-ik_x r} C_r(k_x, k_y) \Rightarrow M^{-1} H' M = H$

where $M = \mathrm{diag}(e^{-ik_x r})$ and $M^{-1} = \mathrm{diag}(e^{ik_x r})$.

$\rightarrow (\mathrm{diag}\, e^{+ik_x r}) \cdot H'(k_x, k_y)\, \mathrm{diag}(e^{-ik_x r}) = \mathrm{diag}(e^{i(k_x + \frac{2\pi}{q})\cdot r})$

$H'(k_x + \frac{2\pi}{q}, k_y)\, \mathrm{diag}(e^{-i(k_x + \frac{2\pi}{q})\cdot r})$

$\Rightarrow \boxed{H'(k_x, k_y) = \mathrm{diag}(e^{i\frac{2\pi}{q}\cdot r})\, H'(k_x + \frac{2\pi}{q}, k_y)\, \mathrm{diag}(e^{-i\frac{2\pi}{q} r})}$

If q is even, we have $H'(k_x + \pi, k_y + \pi) = -H'(k_x, k_y)$

On the other hand, $H'(k_x, k_y) \sim H'(k_x + \pi, k_y) \sim H'(k_x + \pi, k_y + \pi)$, (from the above two symmetry operations).

This means that for q even, $H'(k_x, k_y) \sim -H'(k_x, k_y)$, Let's denote $H'(k_x, k_y) = -\Gamma^{-1} H'(k_x, k_y)\, \Gamma$

$\Rightarrow \boxed{\{H'(k_x, k_y), \Gamma\} = 0. \text{ and } \Gamma^2 = I}$

$\boxed{\text{HW: Check that } \Gamma = (i)^{q/2}\, \mathrm{diag}(e^{-i\pi r})\, T^{q/2}, \text{ that } \Gamma^2 = I.}$

Proof: $H'(k_x, k_y) = T^{-1} H'(k_x, k_y - 2\pi p/q) T = (T^{-1})^{q/2} H'(k_x - \pi p,) T^{q/2}$

$= (T^{-1})^{q/2} H'(k_x, k_y + \pi) T^{q/2}$

$H'(k_x, k_y) = \mathcal{U}^{-1} H'(k_x + \frac{2\pi}{q}, k_y) \mathcal{U}$, with $\mathcal{U} = \mathrm{diag}\, e^{-i\frac{2\pi}{q} r}$.

$= (\mathcal{U}^{-1})^{q/2} H'(k_x + \pi, k_y) \mathcal{U}^{q/2}$

$\Rightarrow H'(k_x, k_y) = (T^{-1})^{q/2} (\mathcal{U}^{-1})^{q/2} H'(k_x + \pi, k_y + \pi) \mathcal{U}^{q/2} T^{q/2}$

$= - \Gamma^{-1} H'(k_x, k_y) \Gamma$, where $\Gamma = i^{q/2} \mathcal{U}^{q/2} T^{q/2}$.

$$\boxed{\Gamma_{ij} = i^{q/2} (-)^{i} \delta_{i+q/2, j}}$$

$\mathcal{U}^{q/2} = \mathrm{diag}(e^{-i\pi r})$

$T^{q/2} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

$\Rightarrow (\Gamma^2)_{ij} = \Gamma_{il} \Gamma_{lj}$

$= i^{q} (-)^{i} \delta_{i+q/2, l} (-)^{l} \delta_{l, j-q/2}$

$= i^{q} (-)^{i+j-q/2} \delta_{i+q/2, j-q/2} = \delta_{ij}$ ← i, j index d are defined modular "q".

Check $(\Gamma \mathcal{U})_{ij} = i^{q/2} (-)^{i} \delta_{i+q/2, j} e^{-i\frac{2\pi}{q} j} = i^{q/2} (-)^{i+1} e^{-i\frac{2\pi}{q} i} \delta_{i+q/2, j}$

$(\mathcal{U}\Gamma)_{ij} = e^{-i\frac{2\pi}{q} i} i^{q/2} (-)^{i} \delta_{i+q/2, j}$

$\Rightarrow \boxed{\{\Gamma, \mathcal{U}\} = 0}$

$(\Gamma T)_{ij} = i^{q/2} (-)^{i} \delta_{i+q/2, l} \delta_{l+1, j} = i^{q/2} (-)^{i} \delta_{i+q/2, j-1}$ ← same

$(T\Gamma)_{ij} = \delta_{i+1, l} i^{q/2} (-)^{l} \delta_{l+q/2, j} = i^{q/2} (-)^{i+1} \delta_{i+1+q/2, j}$

$\Rightarrow \boxed{\{\Gamma, T\} = 0}$.

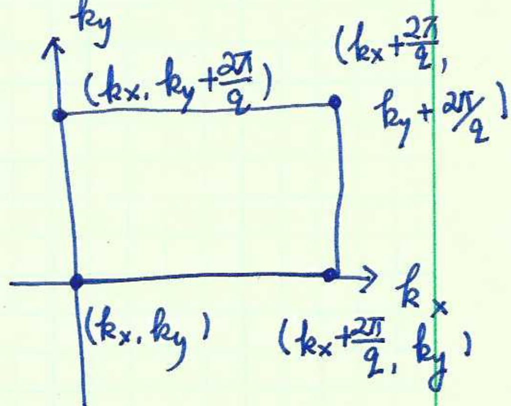
HW: Prove that in the diagonal Rep of Γ, we have

$$\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad H'(k_x, k_y) = \begin{pmatrix} 0, & h^+ \\ h & 0 \end{pmatrix}$$

and $\det H'(k_x, k_y) = \mp |\det h|^2$, where "+" for $q = 4n$ and "−" for $q = 4n+2$. Hint: expand $\det H'(k_x, k_y)$ in terms of the definition of determinant.

In the neighbourhood of $\vec{k}$, if $D = \det h \neq 0$, we define

$A_i = \frac{\partial}{\partial k_i} \ln D$, and around a loop C in the magnetic BZ, we define $\nu = \frac{1}{2\pi i} \oint_C dk_i \frac{\partial}{\partial k_i} \ln D$. If $\nu \neq 0$, then $D(k) = \det h(\vec{k})$ must have zeros inside C. ν = # positive vortex − # negative vortex.



~~Acc~~

Since $\{\Gamma, T\} = 0$, if $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$,

T can be expressed as $T = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}$.

Then $H'(k_x, k_y) = T^{-1} H'(k_x, k_y - 2\pi p/q)\, T$

$$\begin{pmatrix} 0 & h^+(k_x,k_y) \\ h(k_x,k_y) & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_2^{-1} \\ T_1^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & h^+(k_x, k_y - 2\pi p/q) \\ h(k_x, k_y - 2\pi p/q) & 0 \end{pmatrix} \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & T_2^{-1} h(\cdots) T_1 \\ T_1^{-1} h^+(\cdots) T & \end{pmatrix}$$

$$\Rightarrow \boxed{h(k_x, k_y) = T_1^{-1}\, h^+(k_x, k_y - 2\pi p/q)\, T_2}$$

Conside $np + mq = -1$, or $np = -1 - mq$. Since p, q coprime and q is even, p is odd and n should also be odd. $\Rightarrow$

$$\boxed{h(k_x, k_y) = T_1^{-n} h^{\dagger}(k_x, k_y + 2\pi/q) T_2^{n}}$$

From $H'(k_x, k_y) = \mathcal{U}^{-1} H'(k_x + \frac{2\pi}{q}, k_y)\, \mathcal{U}$. In the basis of $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$

$$\mathcal{U} = \begin{pmatrix} 0 & \mathcal{U}_1 \\ \mathcal{U}_2 & 0 \end{pmatrix} \Rightarrow \boxed{h(k_x, k_y) = \mathcal{U}_1^{-1} h^{\dagger}(k_x + 2\pi/q, k_y)\, \mathcal{U}_2}$$

Now let's consider the relation between $\mathcal{U}$ and T.

$$(\mathcal{U} T)_{ij} = e^{-i \frac{2\pi}{q} i} \delta_{i+1,j}$$

$$(T \mathcal{U})_{ij} = \delta_{i+1,j}\, e^{-i \frac{2\pi}{q} j} = \left(e^{-i\frac{2\pi}{q} i} \delta_{i+1,j}\right) e^{-i 2\pi/q}$$

$$\Rightarrow \mathcal{U} T = T \mathcal{U} e^{-i 2\pi/q} \Rightarrow \begin{pmatrix} \mathcal{U}_1 T_2 & 0 \\ 0 & \mathcal{U}_2 T_1 \end{pmatrix} = \begin{pmatrix} T_1 \mathcal{U}_2 & 0 \\ 0 & T_2 \mathcal{U}_1 \end{pmatrix} e^{-i 2\pi/q}$$

$$\Rightarrow \mathcal{U}_1 T_2 = T_1 \mathcal{U}_2\, e^{-i 2\pi/q} \Rightarrow \mathcal{U}_1 T_2^n = T_1^n \mathcal{U}_2\, e^{-i 2\pi n/q}.$$

$$\mathcal{U}_2 T_1 = T_2 \mathcal{U}_1\, e^{-i 2\pi/q} \qquad \mathcal{U}_2 T_1^n = T_2^n \mathcal{U}_1\, e^{-i 2\pi n/q}$$

$$\Rightarrow \det[\mathcal{U}_1 T_2^n] = \det[T_1^n \mathcal{U}_2]\, e^{-i 2\pi n/q \cdot \frac{q}{2}} = \det[T_1^n \mathcal{U}_2]\, e^{-i n\pi}$$

$$= -\det[T_1^n \mathcal{U}_2] \Rightarrow \det[\mathcal{U}_1 \mathcal{U}_2^{-1}] = -\det[T_1^n T_2^{-n}]$$

$$\text{or } \boxed{\det[\mathcal{U}_1^{-1} \mathcal{U}_2] = -\det[T_1^{-n} T_2^{n}]}$$

$$\Rightarrow \quad \det h(k_x, k_y) = \det h^{\dagger}(k_x, k_y + 2\pi/q) \det [T_2^n T_1^{-n}]$$

$$\det h(k_x, k_y) = \det h^{\dagger}(k_x + 2\pi/q, k_y) \det [\mathcal{U}_1^{-1} \mathcal{U}_2]$$

It's easy to show that since $\mathcal{U}$ and T are unitary matrices, all T_1, T_2 and $\mathcal{U}_1, \mathcal{U}_2$ are also unitary. Thus $\det[T_2^n T_1^{-n}]$ and $\det[\mathcal{U}_1^{-1}\mathcal{U}_2]$ are just a unitary phase. Let's denote $\det[T_2^n T_1^{-n}] = e^{i\theta}$ then $\det[\mathcal{U}_1^{-1}\mathcal{U}_2] = -e^{i\theta}$, where θ is a const phase independent on k.

Then we have $D(k_x, k_y) = D^*(k_x, k_y + 2\pi/q)\, e^{i\theta} = -D^*(k_x + 2\pi/q, k_y)\, e^{i\theta}$

we also have $\det h^{\dagger}(k_x, k_y + \frac{2\pi}{q}) = \det h^{\dagger}(k_x + 2\pi/q, k_y + 2\pi/q)\,(-e^{i\theta})$

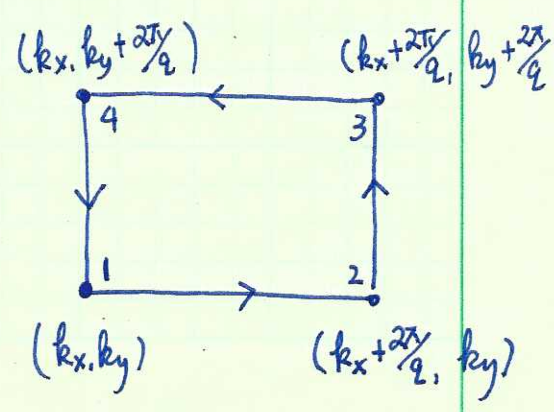
$$\Rightarrow \det h(k_x, k_y) = \det h^{\dagger}(k_x, k_y + 2\pi/q)\, e^{i\theta}$$
$$= -\det h(k_x + 2\pi/q, k_y + 2\pi/q)$$

Sumarize:

$$\boxed{D(k_x, k_y) = -D^*(k_x + 2\pi/q, k_y)\, e^{i\theta} = -D(k_x + 2\pi/q, k_y + 2\pi/q) = D^*(k_x, k_y + 2\pi/q)\, e^{i\theta}}$$

Since θ is a const independent of (k_x, k_y) we set $\theta = 0$, without changing the topo-index

$$\int_1^2 dk_x\, \partial_{k_x} \ln D(k_x, k_y) + \int_3^4 dk_x\, \partial_{k_x} \ln D(k_x, k_y)$$



$$= \int_1^2 dk_x \ \partial_{k_x} [\ln D(k_x, k_y) - \ln D(k_x, k_y + \tfrac{2\pi}{q})] \leftarrow \overset{*}{D}(k_x, k_y)$$

$$= \int_1^2 dk_x \, 2 \, \partial_{k_x} \ln D(k_x, k_y)$$

$$\left[\int_2^3 dk_y + \int_4^1 dk_y \ \partial_{k_y} \ln D(k_x, k_y)\right] = \int_2^3 dk_y \ \partial_{k_y} [\ln D(k_x, k_y) - \ln D(k_x - \tfrac{2\pi}{q}, k_y)]$$

$$D(k_x - \tfrac{2\pi}{q}, k_y) = -D^*(k_x, k_y) e^{i\theta} \rightarrow \partial_{k_y} \ln D(k_x - \tfrac{2\pi}{q}, k_y) = \partial_{k_y} \ln D(k_x, k_y)$$

$$\Rightarrow \quad 2 \int_2^3 dk_y \ \partial_{k_y} \ln D(k_x, k_y)$$

$$\Rightarrow \oint d\vec{k} \cdot \partial_{\vec{k}} \ln D = 2 \left[\int_1^2 + \int_2^3 d\vec{k} \cdot \partial_k \ln D\right]$$

since $D(k_x + \frac{2\pi}{q}, k_y + \frac{2\pi}{q}) = -D(k_x, k_y) \Rightarrow \ln D_k(k_x + \frac{2\pi}{q}, k_y + \frac{2\pi}{q})$

$= \ln D(k_x, k_y) + (2m+1)\pi i$

$$\Rightarrow \boxed{\nu = \frac{1}{2\pi i} \oint d\vec{k} \cdot \partial_{\vec{k}} \ln D = (2m+1)}$$

Thus $\nu \neq 0$, there must be zeros inside the loop.

⊛ Next we need to determine the location of Dirac points

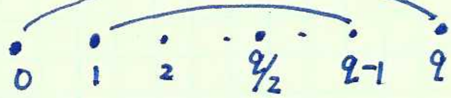
Recall

$$H'(q_x + \tfrac{\pi}{2}, q_y + \tfrac{\pi}{2}) = \begin{bmatrix} 2t_y \sin(q_y - 2\pi\phi), & -it_x e^{iq_x}, & \cdots & it_x e^{-iq_x} \\ it_x e^{-iq_x}, & 2t_y \sin(q_y - 4\pi\phi), & -it_x e^{iq_x}, \cdots & 0 \\ & & & \\ -it_x e^{iq_x}, & \cdots & & it_x e^{-iq_x} + 2t_y \sin q_y \end{bmatrix}$$

$$H'(q_x+\tfrac{\pi}{2}, q_y+\tfrac{\pi}{2}) = 2t_y \sin(q_y - 2\pi i \phi)\,\delta_{ij} - it_x e^{iq_x}\,\delta_{i+1,j} + it_x e^{-iq_x}\,\delta_{i-1,j}$$

compare $H'(-q_x+\frac{\pi}{2}, -q_y+\pi/2) = -2t_y \sin(q_y + 2\pi i\phi)\,\delta_{ij} - it_x e^{-iq_x}\,\delta_{i+1,j}$
$\qquad + it_x e^{iq_x}\,\delta_{i-1,j}$

define P : inversion



$$P_{ij} = \delta_{q-i,j} \Rightarrow P^2 = 1$$

$$[P\,H'(q_x+\tfrac{\pi}{2}, q_y+\tfrac{\pi}{2})\,P]_{ij} = P_{q-i,l}\; H'_{lm}(q_x+\tfrac{\pi}{2}, q_y+\tfrac{\pi}{2})\; P_{mj}$$

$$= \delta_{q-i,l}\,[2t_y \sin(q_y - 2\pi l\phi)\,\delta_{lm} - it_x e^{iq_x}\,\delta_{l+1,m} + it_x e^{-iq_x}\,\delta_{l-1,m})\,\delta_{q-m,j}$$

$\uparrow$ index

$$= 2t_y \sin(q_y - 2\pi(q-i)\tfrac{p}{q})\,\delta_{q-i,q-j} - it_x e^{iq_x}\,\delta_{q-i+1,q-j} + it_x e^{-iq_x}\,\delta_{q-i-1,q-j}$$

$$= 2t_y \sin(q_y + 2\pi i\phi)\,\delta_{ij} + it_x e^{-iq_x}\,\delta_{i+1,j} - it_x e^{iq_x}\,\delta_{i-1,j} = -H'(-q_x+\tfrac{\pi}{2}, -q_y+\tfrac{\pi}{2}).$$

$\uparrow$ site index

$$\Rightarrow \boxed{P\,H'(q_x+\tfrac{\pi}{2}, q_y+\tfrac{\pi}{2})\,P = -H'(-q_x+\tfrac{\pi}{2}, -q_y+\tfrac{\pi}{2})}$$

at $q_x = q_y = 0 \Rightarrow P\,H'(\frac{\pi}{2},\frac{\pi}{2})\,P = -H'(\frac{\pi}{2},\frac{\pi}{2})$, i.e. Chiral sym.

The zero energy states can be chousen as P's eigen states.

Assum n_+ and n_- are numbers of zero energy states of $H'(\frac{\pi}{2},\frac{\pi}{2})$ with $P = \pm 1$. Define chiral index I

$$I = n_+ - n_- = \mathrm{Tr}_{E=0}\, P.$$

For nonzero energy states, P maps a state of energy E to another one with the energy $-E$, thus P is off-diagonal.

$$\Rightarrow \boxed{\mathrm{Tr}\,P = \mathrm{Tr}_{E\neq 0}\,P + \mathrm{Tr}_{E=0}\,P = I = n_+ - n_-}$$

$P_{ij} = \delta_{q-i,j}$ for q even. we have $P_{qq} = P_{q/2\,q/2} = 1 \Rightarrow \mathrm{Tr}\,P = 2$.

$\Rightarrow H'(\frac{\pi}{2},\frac{\pi}{2})$ at least has two zero-eigenstates. We denot $k^* = (\frac{\pi}{2},\frac{\pi}{2})$ below.

* Another definition of index — independent on basis

$$\nu = \frac{1}{4\pi i}\oint_C tr\left(\Gamma H^{-1} dH\right)$$

around C, H has no zero mode such that H^{-1} is well-defined.

$$= \frac{1}{4\pi i}\oint tr\left[h^{-1}dh - h^{+-1}dh^{+}\right]$$

define $h = D\tilde{h}$ where $D = \det h$, and $\det\tilde{h} = 1$

$$h^{-1}dh = D^{-1}\tilde{h}^{-1}d(D\tilde{h}) = D^{-1}dD + \tilde{h}^{-1}d\tilde{h}$$

$$tr\left[\tilde{h}^{-1}d\tilde{h}\right] = d\left(tr\ln\tilde{h}\right) = d\left[\ln\det\tilde{h}\right] = 0$$

$$h^{+-1}dh^{+} = D^{*-1}dD^{*} \Rightarrow \nu = \frac{1}{4\pi i}\oint_\nu\left[d\ln D - d\ln D^{*}\right]$$

$$= \frac{1}{2\pi i}\oint_\nu d\ln D$$

(The trace need to be properly normalized).

Now we define $\nu_1 = \frac{1}{4\pi i}\oint_{C_1} tr\left[\Gamma H^{-1}dH\right]$ for C_1 a small circle centered at $k^* = (\pi/2, \pi/2)$.

$\leftarrow$ Please check $\boxed{\Gamma P = \pm P \Gamma}$ + for q even, − for q odd

we have $P H'(\vec{k}+\vec{k}^*) P = - H'(-\vec{k}+\vec{k}^*)$

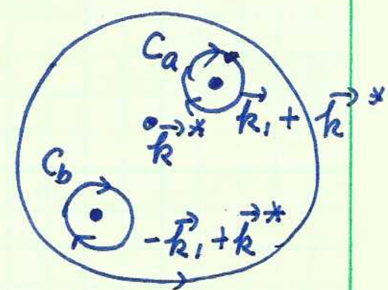
if $\vec{k}_1+\vec{k}^*$ is a zero, then $-\vec{k}_1+\vec{k}^*$ is also a zero

$$\oint_{C_a} d\vec{k}\ \mathrm{tr}\left[\Gamma\ H'^{-1}(\vec{k}+\vec{k}^*)\, d\, H'(\vec{k}+\vec{k}^*) \right]$$

$\mathrm{tr}[P\Gamma H'^{-1} d H' P] = \mathrm{tr}[\Gamma (P H'^{-1} P) d (P H' P)]$

since $P\Gamma = \Gamma P$

$$= \oint_{C_a} d\vec{k}\ \mathrm{tr}\left[\Gamma\ H'(-\vec{k}+\vec{k}^*)\, d\, H'(-\vec{k}+\vec{k}^*) \right)$$



check $\vec{k}\to-\vec{k}$, $C_a \to C_b$, but the orientation does not change

$$\Rightarrow \oint_{C_b} d\vec{k}\ \mathrm{tr}\left[\Gamma\ H'(\vec{k}+\vec{k}^*)\, d\, H'(\vec{k}+\vec{k}^*) \right]$$

$\Rightarrow \pm\vec{k}_1+\vec{k}^*$ are zero's with same winding #.

* Another property $\boxed{P H'(k_x, k_y) P = H'(-k_x, -k_y)}$

$$H'(k_x, k_y) = -2t_y \cos(k_y - 2\pi i \phi)\delta_{ij} - t_x e^{ik_x}\delta_{i+1,j} - t_x e^{-ik_x}\delta_{i-1,j}$$

$$P_{ij} = \delta_{q-i,j}$$

$$[P H'(k_x,k_y) P]_{ij} = P_{i,l}\, H'_{lm}(k_x,k_y)\, P_{mj}$$

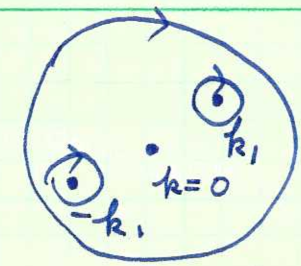
$$= \delta_{q-i,l}\left[-2t_y\cos(k_y-2\pi i\phi)\delta_{lm} - t_x e^{ik_x}\delta_{l+1,m} - t_x e^{-ik_x}\delta_{l-1,m}\right)\delta_{q-m,j}$$

$$= -2t_y\cos(k_y - 2\pi(q-i)\phi)\delta_{ij} - t_x e^{ik_x}\delta_{q-i+1,q-j} - t_x e^{-ik_x}\delta_{q-i-1,q-j}$$

$$= -2t_y\cos(k_y + 2\pi i\phi)\delta_{ij} - t_x e^{-ik_x}\delta_{i+1,j} - t_x e^{ik_x}\delta_{i-1,j}$$

$$= H(-k_x, -k_y)$$

for a circle C_0 centered at $k=0$.



We also define $\nu_0 = \frac{1}{4\pi i}\oint_{C_0} \mathrm{tr}\left[\Gamma \mathcal{H}^{-1} d\mathcal{H}\right]$.

Again since $P H'(k_x, k_y) P = H'(-k_x, -k_y)$,

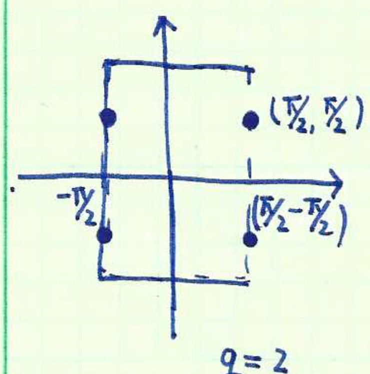
if k_1 is a zero, then $-k_1$ is also a zero with the same winding #.

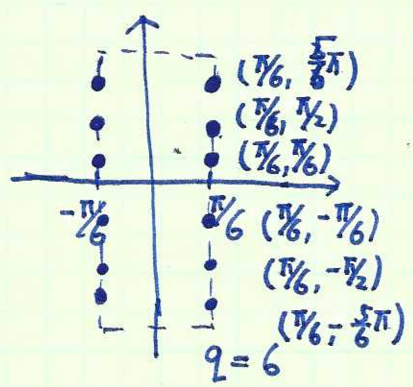
Thus $\underline{\nu_0 \bmod Z_2}$ and $\underline{\nu_1 \bmod Z_2}$ are topological invariants.

Below we will prove for

$$\boxed{\begin{array}{ll} q = 4n+2, & e^{i\pi\nu_0} = 1, \quad e^{i\pi\nu_1} = -1 \\ q = 4n & e^{i\pi\nu_0} = e^{i\pi\nu_1} = -1. \end{array}}$$

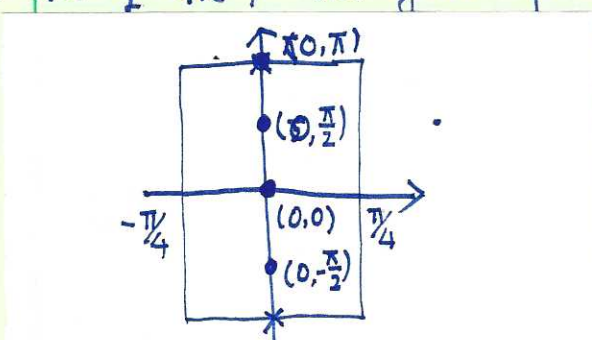
☆ The Dirac point pattern can be summarized as starting from $\vec{k}^* = (\pi/2, \pi/2)$ and $\vec{k}^* + \frac{2\pi}{q}(m_x, m_y)$. ① For $q = 4n+2$, the grid of Dirac points can be represented as $\frac{\pi}{4n+2}(2l_x+1, 2l_y+1)$.





← $\vec{k} = 0$ is not a zero.

② For $q = 4n$, the grid of Dirac points $\frac{\pi}{2n}(l_x, l_y)$



- Next let us prove these results

We consider the limit of $t_x/t_y = \tau \to 0$, and treat the hopping along x-direction as a perterbation. Consider $\vec{k} = \vec{k}^* + \Delta\vec{k}$, with $\vec{k}^* = (\pi/2, \pi/2)$

check $H'(\vec{k}^* + \Delta\vec{k})$: look at the diagonal term $-2t_y \cos(k_y - 2\pi r \, p/q)$.

At $r = q/2$ and q, these two terms are close to zero.

$$\begin{cases} \cos(\pi/2 + \Delta k_y - p\pi) = \sin(\Delta k_y) \\ \cos(\pi/2 + \Delta k_y - 2p\pi) = -\sin\Delta k_y \end{cases} \Rightarrow \underbrace{\Delta H = \sin\Delta k_y \, \sigma_z}_{(\Delta k_x = 0, \Delta k_y)} \leftarrow$$ reduce to two-level problem.

Then add $\Delta k_x \neq 0$, as $\tau \to 0$, according to perturbation theorem, from $q/2$ to q, there are two independent paths

$$\begin{cases} q/2 \to q/2 + 1 \to \cdots \to q-1 \to q \quad ① \\ q/2 \to q/2 - 1 \to \cdots \to 1 \to q \quad ② \end{cases}$$

For either path, there're $q/2 - 1$ intermediate high energy states. The energy denominators at each ~~stat~~ step $q/2 \pm l$:

$$\begin{cases} -2t_y \sin\left((q/2 + l)\, 2\pi \, p/q\right) = 2t_y \sin(l \, 2\pi \, p/q) \\ -2t_y \sin\left((q/2 - l)\, 2\pi \, p/q\right) = -2t_y \sin(l \, 2\pi \, p/q) \end{cases}$$

they're opposite to each other.

For $q = 4n+2$: we have matrix element

$$\frac{(t_x e^{ik_x})^{q/2}}{\prod_{l=1}^{q/2-1} 2t_y \sin l\, 2\pi \, p/q} + \frac{(t_x e^{-ik_x})^{q/2}}{\cdots} \propto \tau^{q/2} \cos\frac{q}{2} k_x$$

$$= \tau^{2n+1} (-)^{n+1} \sin\Delta k_x$$

$q = 4n$: the matrix element

$$\frac{(t_x e^{ik_x})^{q/2}}{\prod_{l=1}^{2n-1} 2t_y \sin l \cdot 2\pi p/q} + \frac{(t_x e^{-ik_x})^{q/2}}{(-)^{2n-1} \prod_{l=1}^{2n-1} 2t_y \sin l \cdot 2\pi p/q} \propto i\tau^{q/2} \sin \tfrac{q}{2} k_x$$

$$= i\, \tau^{2n} (-)^n \sin \Delta k_x$$

$\Rightarrow$ The reduced two energy band Hamiltonian

$$\Delta H(\vec{k}^* + \Delta \vec{k}) = \sin \Delta k_y\, \sigma_z + K_q \sin \Delta k_x\, \sigma_x \sim \Delta k_y \sigma_z + K_q \Delta k_x\, \sigma_x$$

$$\begin{cases} \text{for } q = 4n+2 \\ \Delta k_y\, \sigma_z + K_q \Delta k_y\, \sigma_y . \end{cases}$$

K_q is a real number

It's clear that

$$\nu_1 = \frac{1}{2\pi i} \oint_{C_1} \frac{d(\Delta k_y \pm K_q i \Delta k_y)}{\Delta k_y \pm K_q i \Delta k_y} = 1 \bmod (Z_2)$$

Check the diagonal

* The winding number of other Dirac node by translation

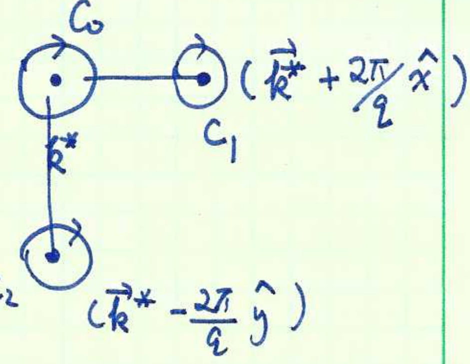
Since $H'(k_x, k_y) = (T^{-1})^n H'(k_x, k_y - 2\pi/q)\, T^n$ — n: odd

$H'(k_x, k_y) = \mathcal{U}^{-1} H'(k_x + \cancel{p}\, 2\pi/q, k_y)\, \mathcal{U}$

also notice that $\{\Gamma, \mathcal{U}\} = \{\Gamma, T\} = 0$, we have

$$\oint_C tr[\Gamma H^{-1} dH] = -\oint_{C_1} tr[\Gamma H'^{-1} dH] = -\oint_{C_2} tr[\Gamma H'^{-1} dH]$$

where C_0 encloses the Dirac nodes at $\vec{k}^*$, and C_1 and C_2 enclose nodes by one $\overbrace{\text{shift}}^{}$ step at $\frac{2\pi}{q}\hat{x}$ or $\frac{2\pi}{q}\hat{y}$.

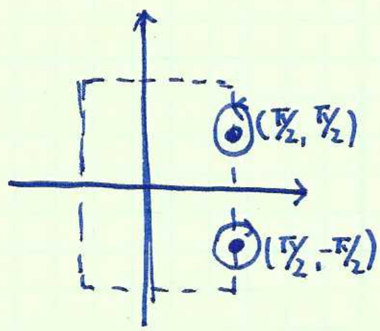


$\boxed{\text{As a result, the winding \#'s of Dirac nodes on page 17 has a staggered pattern.}}$

(*) For example, $q=2$, $h = 2(\cos k_x + i\cos k_y)$

$\sim -(\Delta q_x + i\Delta q_y)$ around $(\pi/2, \pi/2)$

$\{ -\Delta q_x + i\Delta q_y$ $(\pi/2, -\pi/2)$

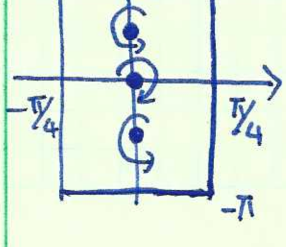


$q=4$: $h = 2\begin{pmatrix} \sin k_y & \cos k_x \\ -i\sin k_x & -\cos k_y \end{pmatrix} \Rightarrow D = -2(\sin 2k_y + i\sin k_x)$

$q=6$ $h = \begin{pmatrix} ia_1 & z^* & z \\ z & ia_2 & z^* \\ z^* & z & ia_3 \end{pmatrix}$ $a_j = 2[\cos k_y + (j-1)\frac{\pi}{3})$

$z = e^{ik_x}$

$\Rightarrow D = \det h = 2(\cos 3k_x + i\cos 3k_y)$



(*) Super symmetric structure

$$Q = \begin{pmatrix} 0 & 0 \\ h & 0 \end{pmatrix} \quad Q^{\dagger} = \begin{pmatrix} 0 & h^{\dagger} \\ 0 & 0 \end{pmatrix} \Rightarrow \mathcal{H} = Q + Q^{2}$$

$$\mathcal{H}^2 = \{Q, Q^{\dagger}\} = \begin{pmatrix} h^{\dagger}h & 0 \\ 0 & hh^{\dagger} \end{pmatrix} \leftarrow \text{supersymmetric}$$

if H^2 has zero modes : $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \rightarrow \begin{cases} h\psi_1 = 0 \\ h^{\dagger}\psi_2 = 0 \end{cases}$