Lect 7 • QHE conductance of a non-interacting system

Let's use the Landau gauge $\vec{A} = -B(y, 0)$, then

$$H = \frac{P_y^2}{2m} + \frac{1}{2} m\omega_c^2 \left(y + \frac{\ell_B^2 P_x}{\hbar}\right)^2 = \frac{P_y^2}{2m} + \frac{(P_x + \frac{e}{c}By)^2}{2m}$$

Defin $\begin{cases} D_z = 2\partial_z + \frac{i}{\ell_B^2} y \\ D_{\bar z} = 2\partial_{\bar z} + \frac{i}{\ell_B^2} y \end{cases}$ $\quad z = x+iy$, $\bar z = x - iy$, $\quad x = \frac{z+\bar z}{2}$, $y = \frac{z-\bar z}{2i}$, $\quad \partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$

Prove: $H = -\frac{\hbar^2}{2m} D_z D_{\bar z} + \text{cnst}$

Proof: $D_z D_{\bar z} = (2\partial_z + \frac{i}{\ell_B^2} y)(2\partial_{\bar z} + \frac{i}{\ell_B^2} y)$

$$= 4\partial_z\partial_{\bar z} + \frac{i}{\ell_B^2} 2(\partial_z y + y\partial_{\bar z}) - \frac{1}{\ell_B^4} y^2$$

$$= (\partial_x^2 + \partial_y^2) + \frac{i}{\ell_B^2} 2y\partial_x - \frac{1}{\ell_B^4} y^2 + \frac{1}{\ell_B^2}$$

$$\Rightarrow \boxed{H = -\frac{\hbar^2}{2m} D_z D_{\bar z} + \frac{1}{2}\hbar\omega}$$

also $[D_z, D_{\bar z}] = \frac{2i}{\ell_B^2}\left\{[\partial_z, y] + [y, \partial_{\bar z}]\right\} = \frac{2i}{\ell_B^2}\left[\frac{-i}{2}[\partial_y, y] + \frac{i}{2}[y, \partial_y]\right] = \frac{-2}{\ell_B^2}$

The ground state satisfies

$$D_{\bar z}\, \psi(x, y) = 0,$$

HW: prove that $\psi(x,y) = f(z)\, e^{-\frac{y^2}{2\ell_0^2}}$

where $f(z)$ is an analytic function.

For infinite system, we have $\psi(x,y) = e^{ikx}\, e^{-\frac{(y+k\ell_B^2)^2}{2\ell_B^2}}$

$$\Rightarrow \boxed{\psi(x,y) \propto e^{ik(x+iy)}\, e^{-\frac{y^2}{2\ell_B^2}}}$$

But this wavefunction does not obey magnetic translation symmetry.

We will construct Landau levels satisfying magnetic Bloch theorem.

Defining the magnetic unit cell $l_x \times l_y$ enclosing a fundametal flux ϕ_0.

For the Landau gauge $\vec{A} = -B(y,0)$, the magnetic translation

$$T_{lx} = e^{-l_x \partial_x}, \quad T_{ly} = e^{-l_y\partial_y - i l_y x/\ell_B^2}$$

plug in $\psi_{k_x k_y}(x,y) = f(z)\, e^{-y^2/2\ell_B^2}$, use $\vec{k} = (k_x, k_y)$

$$\Rightarrow T_{lx}\psi_{\vec{k}}(x,y) = \psi(x-l_x, y) = e^{-ik_x l_x}\psi(x,y)$$

$$f_{k_1k_2}(z-l_x) = e^{-ik_x l_x} f_{k_1k_2}(z) \Rightarrow \boxed{f_{\vec{k}}(z+l_x) = e^{ik_x l_x} f_{\vec{k}}(z)}$$

$$T_{ly}\psi_{\vec{k}}(x,y) = e^{-il_y x/\ell_B^2}\psi(x, y-l_y) = e^{-ik_y l_y}\psi(x,y)$$

$$\Rightarrow f_{\vec{k}}(z-il_y)\, e^{-\frac{(y+l_y)^2}{2\ell_B^2}} = e^{il_y x/\ell_B^2 - ik_y l_y} f_{\vec{k}}(z)\, e^{-\frac{y^2}{2\ell_B^2}}$$

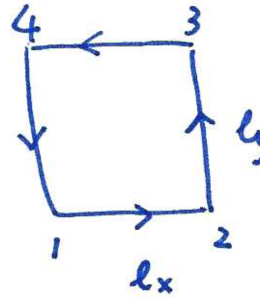
$$f_{\vec{k}}(z+il_y)\, e^{-\frac{(y-l_y)^2}{2\ell_B^2}} = e^{-il_y x/\ell_B^2 + ik_y l_y} f_{\vec{k}}(z)\, e^{-y^2/2\ell_B^2}$$

$$\Rightarrow f_{\vec{k}}(z+il_y) = e^{ik_y l_y}\, e^{-i\frac{l_x l_y}{\ell_B^2}\left[\frac{x+iy}{l_x} + i\frac{l_y}{2l_x}\right]} f_{\vec{k}}(z)$$

since $\ell_B^2 = \frac{\hbar c}{|eB|}$, $\ell_x \ell_y = \frac{\phi_0}{B} = \frac{hc}{eB} \Rightarrow \frac{\ell_x \ell_y}{\ell_B^2} = 2\pi$

$$\Rightarrow \boxed{f_{\vec{k}}(z + i\ell_y) = e^{i k_y \ell_y} e^{-i\pi \left[\frac{2z}{\ell_x} + \tau \right]} f_{\vec{k}}(z)}$$

where $\tau = i\,\ell_y/\ell_x$ is the modular factor.



$$N_\phi = \oint_\gamma \frac{dz}{2\pi i} \frac{f'(z)}{f(z)} = \frac{1}{2\pi i} \oint d \ln f$$

$$= \frac{1}{2\pi i} \left[\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 d \ln f \right]$$

$$= \frac{1}{2\pi i} \left[\int_1^2 \{ \ln f(z) - \ln [f(z + i\ell_y)] \} + \int^4 d [\ln f(z + \ell_x) - \ln f] \right]$$

$$\ln \frac{f(z+\ell_x)}{f} = i k_x \ell_x \Rightarrow d \ln \left(\frac{f(z+\ell_x)}{f} \right) = 0$$

$$\ln \frac{f(z+i\ell_y)}{f} = i k_y \ell_y - i\pi \frac{2z}{\ell_x} + \ell_y/\ell_x$$

$$d \ln \frac{f(z+i\ell_y)}{f} = i\pi 2 \frac{dx}{\ell_x}$$

$$\Rightarrow N_\phi = \frac{1}{2\pi i} \cdot 2\pi i \int_1^2 dx/\ell_x = 1$$

$\Rightarrow$ f(z) has one node in the magnetic unit cell!

We define $f(z) = e^{ikz}\,\theta_1\left(\frac{z-z_0}{l_x}\middle|\tau\right)$, where k and z_0 to be determined. Here θ_1 is the Jocabi θ-function

$$\theta_1(u|\tau) = -i\sum_{n=-\infty}^{\infty}(-)^n q^{(n+1/2)^2} e^{i(2n+1)\pi u}$$

$$= 2\sum_{n=0}^{\infty}(-)^n q^{(n+1/2)^2}\sin(2n+1)\pi u\,, \quad \text{with } q = e^{i\pi\tau} = q^{-\pi l_y/l_x}.$$

θ_1 satisfies $\begin{cases}\theta_1(u+1) = -\theta_1(u) \\ \theta_1(u+\tau) = -N\,\theta_1(u)\end{cases}$ with $N = q^{-1}e^{-2\pi u i}$

$\theta_1(u)$ zeros are at $m + m'\tau$. (single zeros)

Then condition $f_{k_x,k_y}(z+l_x) = e^{ik_xl_x} f_{k_x,k_y}(z) \Rightarrow$

$$\Rightarrow e^{ikl_x}\theta_1\left(\frac{z-z_0}{l_x}+1\middle|\tau\right) = e^{ik_xl_x}\theta_1\left(\frac{z-z_0}{l_x}\middle|\tau\right) \Rightarrow \boxed{e^{ik_xl_x} = -e^{ikl_x} \quad ①}$$

$$f_{k_xk_y}(z+il_y) = e^{ik_yl_y}e^{-i\pi\left[\frac{2z}{l_x}+\tau\right]} f_{k_xk_y}(z)$$

$$e^{-kl_y}\,\theta\left(\frac{z-z_0}{l_x}+\tau\middle|\tau\right) = e^{ik_yl_y - i\pi\left(\frac{2z}{l_x}+\tau\right)}\,\theta\left(\frac{z-z_0}{l_x}\middle|\tau\right)$$

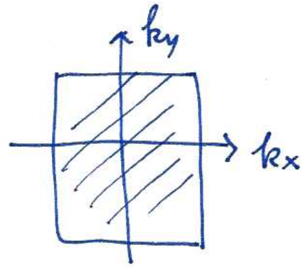
$$-e^{-kl_y}\,\cancel{e^{-i\pi\tau}}\,e^{-2\pi i\left(\frac{\cancel{z}-z_0}{l_x}\right)} = e^{ik_yl_y - i\cancel{\pi\frac{2z}{l_x}} - \cancel{i\pi\tau}}$$

$$\Rightarrow \boxed{e^{ik_yl_y} = -e^{-kl_y + 2\pi i\frac{z_0}{l_x}} \quad ②}$$

From ① $\Rightarrow$ $kl_x = k_xl_x + \pi \quad \Rightarrow \quad \boxed{k = k_x + \pi/l_x}$

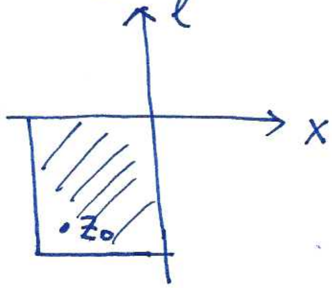
$k_yl_y = 2\pi\frac{z_0}{l_x} + \pi + ikl_y \quad \Rightarrow \quad z_0 = \frac{l_xl_y[k_y - ik_x]}{2\pi} - \left[\frac{l_x + il_y}{2}\right]$

For $k_x \in [-\frac{\pi}{l_x}, \frac{\pi}{l_x}]$, $k_y \in [-\frac{\pi}{l_y}, \frac{\pi}{l_y}]$ $\Rightarrow$



$\Rightarrow Z_0 \in 0, -l_x, -il_y, -l_x - il_y$

Thus for an arbitany Z_0 $\rightarrow$



inside the magnetic unit

cell, there exist a $\vec{k} = (k_x, k_y)$, such at $f_{k_x k_y}(Z_0) = 0$.

or express $k_x l_x = \theta_x$, and $k_y l_y = \theta_y$

we have $\boxed{k = \frac{\theta_x}{l_x} + \frac{\pi}{l_x}, \quad Z_0 = \frac{l_x \theta_y - il_y \theta_x}{2\pi} - \frac{l_x + il_y}{2}}$

Acording to $\psi_{k_x k_y}(x, y) = e^{ik_x x + ik_y y} u_{k_x k_y}(x, y)$

$$= f_{k_x k_y}(z)\, e^{-y^2/2l_B^2}$$

$\Rightarrow$ $u_{k_x k_y}(x, y) = N e^{-i(k_x x + k_y y) - y^2/2l_B^2} f_{k_x k_y}(z)$ where N is normalization factor

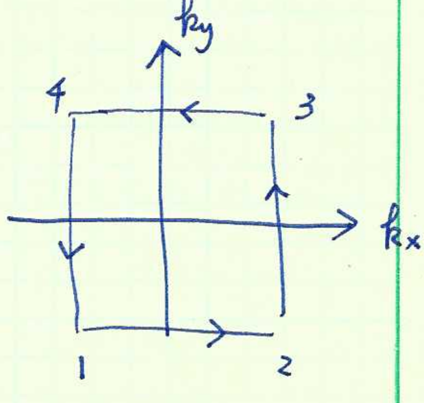
$$A_\alpha(\vec{k}) = \int_{\text{unit cell}} d\vec{r}\; u^*_{k_x k_y}(x, y)\, i \nabla_{k_\alpha} u_{k_x k_y}(x, y)$$

since $e^{ik_x x + ik_y y}$ is a pure phase

$$\nabla_k \times \left(e^{-i(k_x x + k_y y)} \nabla_k e^{ik_x x + ik_y y} \right) = 0$$

We use $A_\alpha(\vec{k}) = N(k) \int_{\substack{\text{unit}\\ \text{cell}}} d\vec{r} \left(f^*_{\vec{k}}\; i\partial_{k_\alpha} f_{\vec{k}} \right) e^{-y^2/2\ell_B^2}$

$$= N(k) \int_{\substack{\text{unit}\\ \text{cell}}} d\vec{r}\; |f_{\vec{k}}(\vec{r})|^2 e^{-y^2/2\ell_B^2}\; i\partial_{k_\alpha} \ln f_{\vec{k}}(\vec{r})$$



Then

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi} \oint d\vec{k} \cdot \vec{A}_{k_\alpha}$$

$$= \frac{e^2}{h}\frac{1}{2\pi} \left[\int_1^2 dk_x \left[A_{k_x}(\vec{k}) - A_{k_x}(\vec{k} + \tfrac{2\pi}{\ell_y}\hat{y})\right] + \int_1^4 dk_y \left[- A_{k_y}(\vec{k}) + A_{k_y}(\vec{k} + \tfrac{2\pi}{\ell_x}\hat{x}\right.\right.$$

since $f_{\vec{k}}(\vec{r})$ and $f_{\vec{k}+\frac{2\pi}{\ell_y}\hat{y}}(\vec{r})$ represents the same state,

they can only differ by an ovell all phase factor $f_{\vec{k}+\frac{2\pi}{\ell_y}\hat{y}}(\vec{r}) = e^{i\Delta\theta_{\vec{k}}} f_{\vec{k}}(\vec{r})$

θ_k is indepedent on $\vec{r}$. then

$$A_{k_x}(\vec{k}) - A_{k_x}(\vec{k} + \tfrac{2\pi}{\ell_y}\hat{y}) = N(\vec{k}) \int_{\substack{\text{unit}\\ \text{cell}}} d\vec{r}\; |f_{\vec{k}}(\vec{r})|^2 e^{-y^2/2\ell_B^2} \overset{\text{independent on } \vec{r}}{\boxed{i\partial_{k_\alpha} \ln \frac{f_{\vec{k}}(\vec{r})}{f_{\vec{k}+\frac{2\pi}{\ell_y}\hat{y}}}}}$$

$$= N(\vec{k})\,(+\Delta\theta_{\vec{k}}) \int_{\substack{\text{unit}\\ \text{cell}}} d\vec{r}\; |f_{\vec{k}}(\vec{r})|^2 e^{-y^2/2\ell_B^2} = +\Delta\theta_{\vec{k}} \;(\text{for } \vec{k} \text{ along } 1\to 2)$$

similarly if we define $f_{\vec{k}+\frac{2\pi}{\ell_x}\hat{x}}(\vec{r}) = e^{i\Delta\theta'_{\vec{k}}} f_{\vec{k}}(\vec{r})$

then $$-A_{k_y}(\vec{k}) + A_{k_y}(\vec{k} + \tfrac{2\pi}{l_x}\hat{x}) = -\Delta\theta_{\vec{k}} = i\,\partial_{k_y} \ln \frac{f_{\vec{k}+\frac{2\pi}{l_x}}(\vec{r})}{f_{\vec{k}}(\vec{r})}$$

$\uparrow$ independent on $\vec{r}$

$\Rightarrow$ overall

$$\sigma_{xy} = \frac{e^2}{h}\frac{(-)}{2\pi i}\left[\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 d\vec{k}\cdot \partial_{\vec{k}} \ln f_{\vec{k}}(\vec{r})\right]$$

we can choose an $\vec{r}$ such that $f_{\vec{k}}(\vec{r}) \neq 0$ for all $\vec{k}$ along the path.

recall $$f_{\vec{k}}(z) = e^{ikz}\,\theta_1\left(\frac{z-z_0}{l_x}\middle|\tau\right)$$
$$= e^{i(k_x+\pi/l_x)z}\,\theta_1\left[\frac{z}{l_x} - \frac{l_y(k_y - ik_x)}{2\pi}\middle|\tau\right]$$

For a given $z \Rightarrow$ $\boxed{k_{0,y} - ik_{0,x} = \dfrac{2\pi z}{l_x l_y}, \text{ which set } f_{\vec{k}_0}(z) = 0}$

as $\vec{k}$ around $\vec{k}_0$, $\dfrac{z-z_0}{l_x}$ around 0 with phase winding 2π.

$\Rightarrow$ $f_{\vec{k}}(z)$ phase also winds 2π since θ_1 has single zero.

$$\Rightarrow \frac{1}{2\pi i}\oint d\vec{k}\;\partial_{\vec{k}} \ln f_{\vec{k}}(z) = \pm 1 \leftarrow$$ vorticity for the wavefunction parameter goes round $\vec{k}_0$

$$\Rightarrow \sigma_{xy} = \pm\, e^2/h$$