Lect 3: Edge mode of 1D topo-superconductor

§: P-wave, Bogoliubov - de Gennes equation

$$H_{MF} = \int dx \; \psi^\dagger \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \mu(x)\right)\psi + \psi^\dagger(x)\frac{\Delta}{k_f}(-i\partial_x)\psi^\dagger(x) + h.c$$

$$= \int dx \begin{bmatrix} \psi^\dagger(x) & \psi(x) \end{bmatrix} \begin{bmatrix} H_0 & \frac{\Delta}{k_f}(-i\partial_x) \\ \frac{\Delta}{k_f}(-i\partial_x) & -H_0 \end{bmatrix} \begin{bmatrix} \psi(x) \\ \psi^\dagger(x) \end{bmatrix}$$

① If $H_0 = -\mu$, then the matrix kerneal $h(x) = -\mu\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{\Delta}{k_f} P_x \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

again, this is the 1D Dirac equation, in the convention $\beta = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

and $\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. However, for the wavefunction $\begin{pmatrix} \psi(x) \\ \psi^\dagger(x) \end{pmatrix}$, it's

upper and lower components are not independent to each other.

Esentially, ~~since~~ it's called Majorana - Dirac Eq.

② we expand

$$\begin{bmatrix} \psi(x) \\ \psi^\dagger(x) \end{bmatrix} = \sum_n \begin{bmatrix} u_n & v_n^* \\ v_n & u_n^* \end{bmatrix} \begin{bmatrix} C_n \\ C_n^\dagger \end{bmatrix} \quad \ast$$

~~for states with real $E_n \neq 0$~~

HW: Prove that if $\begin{bmatrix} H_0 & \frac{\Delta}{k_f}(-i\partial_x) \\ \frac{\Delta}{k_f}(-i\partial_x) & -H_0 \end{bmatrix}\begin{bmatrix} u_n \\ v_n \end{bmatrix} = E_n \begin{bmatrix} u_n \\ v_n \end{bmatrix}$

then $\underbrace{\begin{bmatrix} H_0 & \frac{\Delta}{k_f}(-i\partial_x) \\ \frac{\Delta}{k_f}(-i\partial_x) & -H_0 \end{bmatrix}}_{h(x)}\begin{bmatrix} v_n^* \\ u_n^* \end{bmatrix} = -E_n \begin{bmatrix} v_n^* \\ u_n^* \end{bmatrix}$

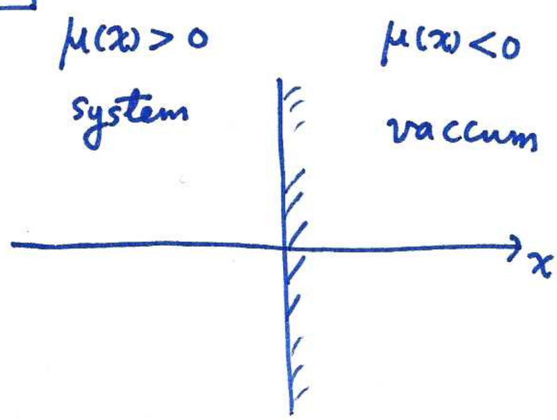
— particle-hole symmetry. Hint: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} h(x) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -h^*(x)$.

$$\Rightarrow \begin{bmatrix} H_0 & \frac{\Delta}{k_f}(-i\partial_x) \\ \frac{\Delta}{k_f}(-i\partial_x) & -H_0 \end{bmatrix}\begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix} = E_n \begin{bmatrix} u_n(x) \\ v_n(x) \end{bmatrix}$$

We seek the zero-energy solution. Again $\tau_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}$ anti-commutes with $h(x)$, i.e. $\{\tau_2, h(x)\} = 0 \Rightarrow$ the zero energy mode can be choosen as τ_2's eigenstate.

$\Rightarrow$ For $E_n = 0$, we have $\boxed{u_0 = \mp i v_0}$.

③ Consider a domain configuration (chemical potentim change sign)

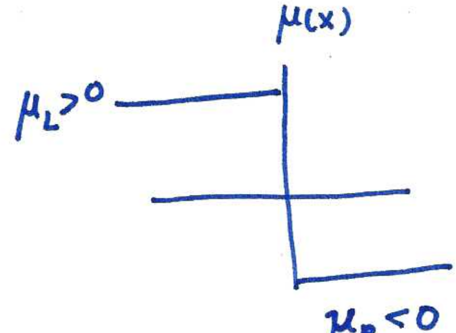


More acuratly, we add the dispersim $H_0 = -\frac{\hbar^2 \partial_x^2}{2m}$

§ Solve zero mode

with $u_0 = -i w_0 \Rightarrow \left[-\frac{\hbar^2 \partial_x^2}{2m} - \mu(x) \right] u_0(x) + \frac{\Delta}{\hbar_f} \partial_x u_0(x) = 0.$

Assume $\mu(x) = \begin{cases} \mu_L > 0 \\ \mu_R < 0 \end{cases}$ step function



$$\Rightarrow \begin{cases} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{\Delta}{\hbar_f} \partial_x \right] u_0 = \mu_L u_0 \\ \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{\Delta}{\hbar_f} \partial_x \right] u_0 = \mu_R u_0 \end{cases}$$

if both μ_L and μ_R are finite, $\Rightarrow u_0''$ is discontinous

but $u_0'(x)$ and $u_0(x)$ remains continous. In order to

Simplify, we take the limit $\mu_R \to -\infty$, i.e. open boundary

condition. $\Rightarrow \begin{cases} u_0'(x) \text{ can be discontinous} \to u_0'(0^+) - u_0'(0^-) = \int_{0^-}^{0^+} dx\, u'' \to \text{finite} \\ u_0(x) \text{ remain contionous} \end{cases}$

boundary condition at $\mu_R \to -\infty$

$$\boxed{u_0(0) = 0, \text{ and } u_0(-\infty) = 0 \leftarrow} \text{ decay solution}$$

we try the solution $u_0 \sim e^{\beta x}$, where β is complex with $Re(\beta) > 0$ to ensure it's decay in the left space. $\Rightarrow$

$$-\frac{\hbar^2\beta^2}{2m} + \frac{\Delta}{k_f}\beta = \mu_L = \frac{\hbar^2 k_f^2}{2m} = \varepsilon_f$$

$$\Rightarrow \left(\beta/k_f\right)^2 - \frac{\Delta}{\varepsilon_f}\left(\beta/k_f\right) + 1 = 0$$

① when discriminate $\left(\frac{\Delta}{\varepsilon_f}\right)^2 - 4 < 0$, consider $\Delta \ll \varepsilon_F$.

~~① when Δ~~ a pair of complex roots

$$\beta/k_f = \frac{1}{2}\frac{\Delta}{\varepsilon_f} \pm i\left[1 - \left(\frac{\Delta}{2\varepsilon_f}\right)^2\right]^{1/2}$$

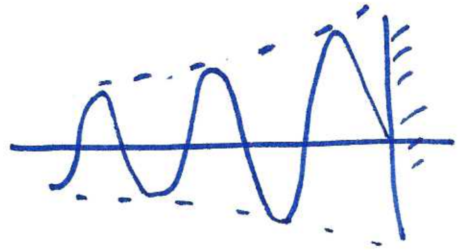
we have $\boxed{u_0(x) \sim e^{\beta_0(x)} \sin \beta_1 x}$

with $\beta_0 = \frac{k_f}{2}\frac{\Delta}{\varepsilon_f}$, $\beta_1 = \sqrt{1 - \left(\frac{\Delta}{2\varepsilon_f}\right)^2}$

$\beta_1 \gg \beta_0$

decay length $1/\beta_0$

oscillating wavector $\sim k_{f_0}$



② If Δ is really strong, $\left(\frac{\Delta}{\varepsilon_f}\right)^2 = 4 \Rightarrow$, then $\beta_0 = k_f$, $\beta_1 = 0$

$$\Rightarrow \quad \boxed{u_0(x) \sim x e^{k_f x}}$$

③ If $\left(\frac{\Delta}{\varepsilon_f}\right)^2 > 4$, $\Rightarrow$ $\frac{\beta_{1,2}}{k_f} = \frac{1}{2}\frac{\Delta}{k_f} \pm \sqrt{\left(\frac{\Delta}{2\varepsilon_f}\right)^2 - 1}$

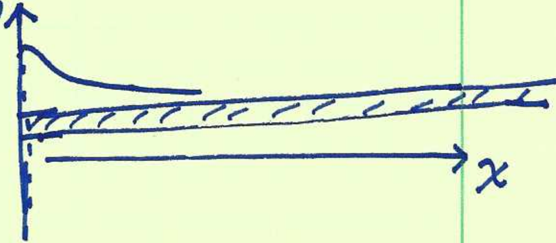
$$\Rightarrow \quad u_0(x) = e^{\beta_1 x} - e^{\beta_2 x} = e^{\beta_2 x}\left[e^{(\beta_1 - \beta_2)x} - 1\right]$$

if $\left(\frac{\Delta}{2\varepsilon_f}\right) \gg 1$, then $u_0(x)$ is domoniate by β_2

in this case, $\beta_1 \sim \Delta$, $\beta_2 \sim \frac{k_f^2}{\Delta}$.

HW: The Majorana nature of the zero energy boundary mode.

Prove that the zero energy mode wavefunction



can be writen as

$$\begin{pmatrix} u_0(x)\, e^{-i\frac{\theta}{2} - i\pi/4} \\ v_0(x)\, e^{i\frac{\theta}{2} + i\pi/4} \end{pmatrix},$$

where θ is the phase of pairing order parameter $\Delta = |\Delta| e^{i\theta}$,

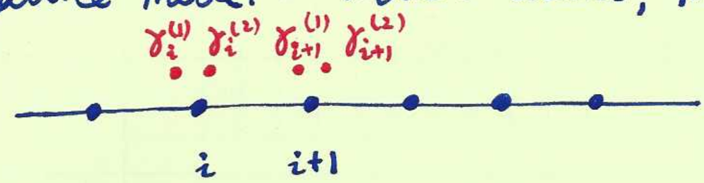
$u_0(x) = v_0(x) \approx e^{-x/\xi} \sin k_f x$ in the limit of $\Delta \ll E_f$.

Please also figure out the expression of ξ. Then the operator associated with this zero mode can be expressed as

$$\boxed{\gamma_0 = \int dx \;\; u_0(x)\, e^{-i\theta/2 - i\pi/4}\, \psi(x) + v_0(x)\, e^{i\theta/2 + i\pi/4}\, \psi^\dagger(x)}$$

then $\gamma_0 = \gamma_0^\dagger$, which is a Majorana fermion operator.

§ lattice model — Kitaev chain, Majorana fermions



$$c_i = \frac{1}{\sqrt{2}}\left(\gamma_i^{(1)} + i\gamma_i^{(2)}\right) \qquad c_i^{+} = \frac{1}{\sqrt{2}}\left(\gamma_i^{(1)} - i\gamma_i^{(2)}\right)$$

Ex: Fermion commutation relation $\Rightarrow \{\gamma_i^{(a)}, \gamma_j^{(b)}\} = 2\delta^{a,b}\delta_{ij}$

$a, b = 1, 2.$

$$H = -\mu \sum_i i\gamma_i^{(1)}\gamma_i^{(2)} + \Delta \sum_i i\gamma_i^{(2)}\gamma_{i+1}^{(1)} \quad \leftarrow \text{Kitaev model}$$

intra site binding inter site binding

$\rightarrow$ a $\sqrt{\ }$ version of the conducting polymer problem.

change back to Dirac fermion representation

$$H = -\mu \sum_i c_i^{+} c_i \ (-)\frac{\Delta}{2} \sum_i \left(c_i^{+} c_{i+1} + c_{i+1}^{+} c_i\right) + \frac{\Delta}{2}\sum_i \left(c_i c_{i+1} - c_i^{+} c_{i+1}^{+}\right)$$

Fourier transform
$$\begin{cases} c_i = \frac{1}{\sqrt{N}} \sum_k e^{ik\cdot i}\, c_k \\ c_i^{+} = \frac{1}{\sqrt{N}} \sum_k e^{-ik\cdot i}\, c_k^{+} \end{cases}$$

$$\Rightarrow H = \sum_k \left[-\mu - \Delta\cos k\right] c_k^{+} c_k - \frac{\Delta}{2}\sum_{k>0}\left(i\sin k\, c_k c_{-k} \cancel{+} - i\sin k\, c_{-k}^{+} c_k^{+}\right)$$

$$= \sum_{k>0} \left(c_k^{+} \ \ c_{-k}\right) \begin{bmatrix} -\mu - \Delta\cos k & \Delta i \sin k \\ -\Delta i \sin k & \mu + \Delta\cos k \end{bmatrix} \begin{bmatrix} c_k \\ c_{-k}^{+} \end{bmatrix}$$

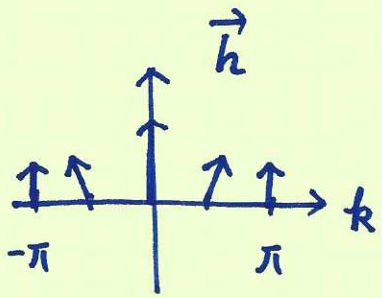
Define $\psi_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} \Rightarrow H = \sum_{k>0} \psi_k^\dagger \, h_k \, \psi_k$

ψ_k and ψ_{-k} not independent $\Rightarrow \boxed{\psi_{-k}^\dagger = \psi_k^T \tau_1}$

and $h_k = -(\mu + \Delta\cos k)\,\tau_3 + \Delta \sin k \,\tau_2$

Anderson's pseudospin

① if $|\mu| > \Delta$, then $\mu + \Delta\cos k$ doesnot change sign

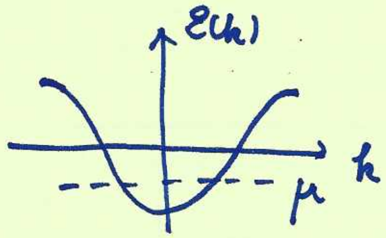


no winding number

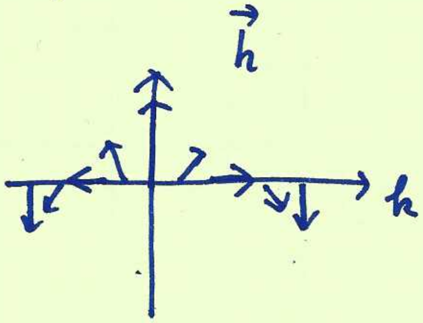
Again define

$Z = (\mu + \Delta\cos k) + i\,\Delta\sin$

$$\omega = \oint \frac{dk}{2\pi i} \frac{1}{Z} \frac{dZ(k)}{dk}$$



② if $|\mu| < \Delta$, then $\mu + \Delta\cos k$ change sign

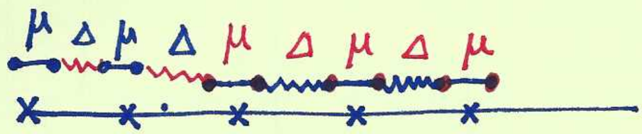


$$\omega = \oint \frac{dk}{2\pi i} \frac{d\ln Z}{dk} = \pm 1$$

Toplogical transitin.

None trivial zero energy boundary mode will appear at $|\mu|<|\Delta$

- Map to an SSH-like mode



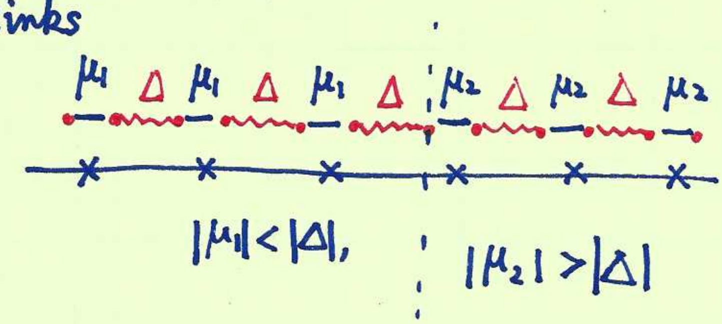
① μ- maps to binding for 2- Majoranas onsite

② Δ - maps to intersite Majorana binding

Since μ always appears on the boundary, the zero energy mode only appear at the boundary if $\mu<\Delta$.

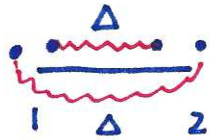
two strong bonds meet → a domain wall

or kinks



Majorana fermion at zero energy!

Conside the case of $\mu = 0$.



Express the bond majurana eigenstate in terms of the on site fermion number occupation. Consider a two site problem. The Hilbert space is $2^2 = 4$ dimensional. According to fermion parity, there are two fermion parity even and two fermion parity odd states.

$$B_1 = i\gamma_1^{(2)}\gamma_2^{(1)} \qquad B_2 = -i\gamma_2^{(2)}\gamma_1^{(1)}$$

and $N_1 = C_1^\dagger C_1 = \dfrac{1 + i\gamma_1^{(1)}\gamma_1^{(2)}}{2}$, $\quad N_2 = C_2^\dagger C_2 = \dfrac{1 + i\gamma_2^{(1)}\gamma_2^{(2)}}{2}$

① Consider $|00\rangle_B$ defined as
$$B_1|00\rangle_B = -|00\rangle_B$$
$$B_2|00\rangle_B = -|00\rangle_B$$

then we start from $|N_1 N_2\rangle_N$ basis, say. $|00\rangle_N$, and perform projection $\bar{P_1} = \dfrac{1 - i\gamma_1^{(2)}\gamma_2^{(1)}}{2}$ $\quad \bar{P_2} = \dfrac{1 + i\gamma_2^{(2)}\gamma_1^{(1)}}{2}$

$$\bar{P_1}\bar{P_2} = \tfrac{1}{4}\left[1 + \gamma_1^{(2)}\gamma_2^{(1)}\gamma_2^{(2)}\gamma_1^{(1)} - i\gamma_1^{(2)}\gamma_2^{(1)} + i\gamma_2^{(2)}\gamma_1^{(1)}\right)$$

$$= \tfrac{1}{4}\left[1 + i\gamma_1^{(1)}\gamma_1^{(2)}\underset{(i)}{}\gamma_2^{(1)}\gamma_2^{(2)} - i(-i)[C_1 - C_1^\dagger][C_2 + C_2^\dagger] + i\underset{(-i)}{}[C_2 - C_2^\dagger][C_1 + C_1^\dagger]\right.$$

$$= \tfrac{1}{4}\left[1 + (2C_1^\dagger C_1 - 1)(2C_2^\dagger C_2 - 1) - [C_1C_2 - C_1^\dagger C_2^\dagger + C_1C_2^\dagger - C_1^\dagger C_2] + [C_2C_1 - C_2^\dagger C_1^\dagger + C_2C_1^\dagger - C_2^\dagger C_1]\right.$$

$$= \tfrac{1}{4}\left[1 + (2C_1^\dagger C_1 - 1)(2C_2^\dagger C_2 - 1)\right] - \tfrac{1}{2}\left[C_1C_2 - C_1^\dagger C_2^\dagger\right)\right]$$

$$P_1 P_2 |00\rangle_N = \tfrac{1}{4}[1+1]|00\rangle_N + \tfrac{1}{2} c_1^\dagger c_2^\dagger |00\rangle_N = \tfrac{1}{2}[|00\rangle_N + |11\rangle_N\rangle$$

Normalization $\quad |00\rangle_B = \frac{1}{\sqrt{2}}[|00\rangle_N + |11\rangle_N]$

Then define $|11\rangle_B = P_1^+ P_2^+ |00\rangle_N = \frac{1+i\gamma_1^{(2)}\gamma_2^{(1)}}{2} \; \frac{1+i\gamma_2^{(2)}\gamma_1^{(1)}}{2} |00\rangle_N$

$$P_1^+ P_2^+ = \tfrac{1}{4}[1+(2c_1^\dagger c_1 - 1)(2c_2^\dagger c_2 - 1) + \tfrac{1}{2}[c_1 c_2 - c_1^\dagger c_2^\dagger]]$$

$$\Rightarrow |11\rangle_B = \frac{1}{\sqrt{2}}[|00\rangle_N - |11\rangle_N]$$

② Define $|10\rangle_B = P_1^+ P_2^- |01\rangle_N = \frac{1+i\gamma_1^{(2)}\gamma_2^{(1)}}{2} \; \frac{1+i\gamma_2^{(1)}\gamma_1^{(2)}}{2} |01\rangle_N$

$$P_1^+ P_2^- = \tfrac{1}{4}[1 - \gamma_1^{(2)}\gamma_2^{(1)}\gamma_2^{(2)}\gamma_1^{(1)} + i\gamma_1^{(2)}\gamma_2^{(1)} + i\gamma_2^{(2)}\gamma_1^{(1)})$$

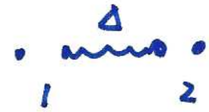
$$= \tfrac{1}{4}[1-(2c_1^\dagger c_1 - 1)(2c_2^\dagger c_2 - 1)] + \tfrac{1}{2}[c_1 c_2^\dagger - c_1^\dagger c_2]$$

$$\Rightarrow |10\rangle_B = \frac{1}{\sqrt{2}}[|01\rangle_N - |10\rangle_N]$$

$$|01\rangle_B = P_1^- P_2^+ |01\rangle_N = \frac{1-i\gamma_1^{(2)}\gamma_2^{(1)}}{2} \; \frac{1-i\gamma_2^{(1)}\gamma_1^{(2)}}{2} |01\rangle_N$$

$$P_1^- P_2^+ = \tfrac{1}{4}[1-(2c_1^\dagger c_1 - 1)(2c_2^\dagger c_2 - 1)] - \tfrac{1}{2}[c_1 c_2^\dagger - c_1^\dagger c_2)$$

$$\Rightarrow |01\rangle_B = \frac{1}{\sqrt{2}}[|01\rangle_N + |10\rangle_N]$$

with open boundary condition 

we have a degeneracy with respect to $B_2 = \pm 1$

$\Rightarrow |10\rangle_B$ and $|11\rangle_B$ degenerate

or $\frac{1}{\sqrt{2}}\left[|01\rangle_N - |10\rangle_N\right]$ and $\frac{1}{\sqrt{2}}\left(|00\rangle_N - |11\rangle_N\right)$