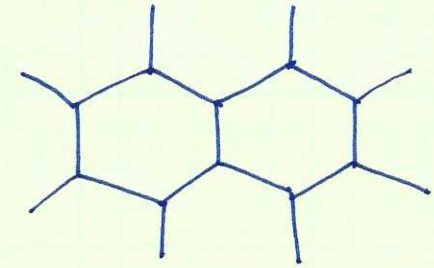
Lect 11 Quantum spin Hall — Z_2 topological insulator

① Kane - Mele model



Consider the honeycomb lattice, with one orbital per site. What's the possible spin-orbit coupling?

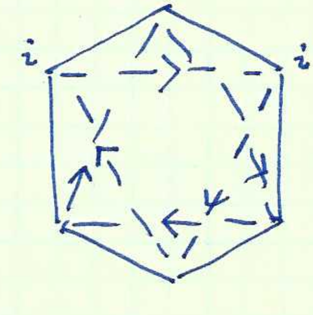
For each nearest-neighbor (NN) bond $\langle ij \rangle$, let's try

$$H_{so} = \sum_{\langle ij \rangle} i\vec{d}_{ij} \cdot \left[c_i^\dagger \vec{\sigma} c_j - c_j^\dagger \vec{\sigma} c_i \right]$$

HW: ① Prove that according to time-reversal symmetry H_{so} has to be written in the form of "spin-current". $\vec{d}_{ij}$ is called the Dzyloshisky-Moriya vector.

② According to the graphene symmetry, $\vec{d}_{ij}$ for NN bond has to vanish.

Now consider the next-nearest-neighbour (NNN) bond $\langle\langle ii' \rangle\rangle$, then

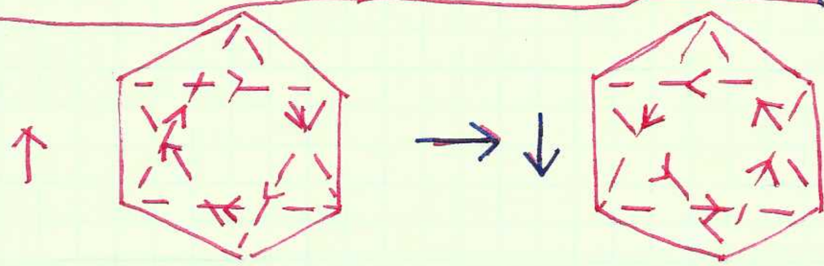


$$H_{so} = \sum_{\langle\langle ii' \rangle\rangle} i\lambda_{so} \left[c_i^\dagger \sigma_z c_{i'} - h.c. \right]$$

HW: ① prove that H_{so} for $\langle\langle ii' \rangle\rangle$ needs to be in the form above i.e. $\vec{d}_{ii'} \parallel \hat{z}$

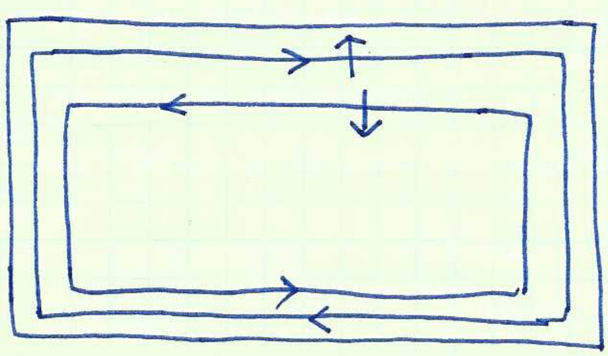
② Prove that the tight-binding model for graphene, when considering spin-orbit coupling, it can be written as a double copy of Haldane model.

i.e
$$H_{KM} = -t\sum_{\langle ij\rangle}\left(C_{i\sigma}^{+}C_{j\sigma} + h.c.\right) - t'\sum_{\langle\langle ii''\rangle\rangle} e^{i\varphi}\left(C_{i\uparrow}^{+}C_{i''\uparrow} - e^{-i\varphi}C_{i''\uparrow}^{+}C_{i\uparrow}\right) + \left[e^{-i\varphi}\, C_{i\downarrow}^{+}C_{i''\downarrow} - e^{i\varphi}C_{i''\downarrow}^{+}C_{i\downarrow}\right]$$



time-reversal double (Kramers double)

(2) helical edge modes



Question: is it stable against impurity scatterings ?

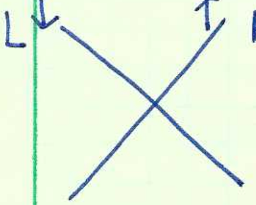
(1) It's not chiral but helical — right mover with spin up left mover with spin down ?

(2) A new one-dimensional state,
  (1) Different from Chiral QHE edge modes (TR breaking
  (2) Different from 1D spinless fermion ($T^2=1$)
  (3) Different from 1D spinful fermion ($T^2=-1$, but two Kramers pair)

③ The helical edge modes cannot be realized in 1D lattice systems. It has to be realized as edge modes in 2D lattices

— C. Wu, B. A. Bernevig, S. C. Zhang, PRL 96, 106401 (2006).

④ For non-interaction case, helical edge modes remain stable

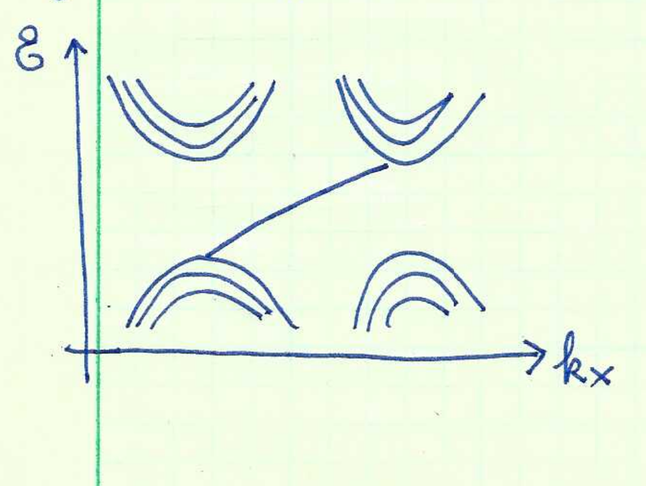


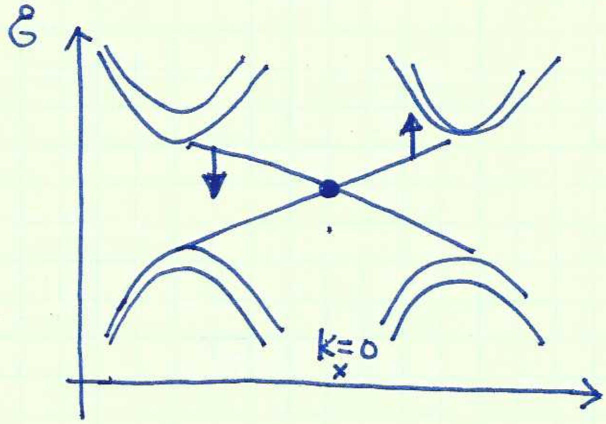
$\Rightarrow$ backscattering

$$H_{bs} = g\left(\psi^\dagger_{R\uparrow}\psi_{L\downarrow} + \psi^\dagger_{L\downarrow}\psi_{R\uparrow}\right)$$

HW: prove that H_{bs} breaks TR symmetry, and thus not allowed!

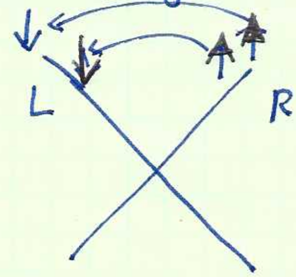
Edge states for Haldane model, Kane-Mele model





The degeneracy point is protected by Kramers symmetry!

- Stability under strong interactions — helical Luttinger liquid



2-particle backscatterings are allowed

$$H_{2\text{-}bs} = g'(\psi^{+}_{R\uparrow}(x)\,\psi^{+}_{R\uparrow}(x+\epsilon)\,\psi_{L\downarrow}(x+\epsilon)\,\psi_{L\downarrow}(x) + h.c.]$$

HW: prove that $H_{2\text{-}bs}$ are time-reversal even, thus are allowed by Kramers (time-reversal) symmetry.

Stability crieterion — Luttinger liquid parameter K

① For coherent UmKlamp scattering, the helical edge states remain stable at $K > 1/2$, and become unstable at $0 < K < 1/2$.

② For an impurity scattering, the helical edge liquid is stable at $K > 1/4$.

When the helical liquid becomes unstable, ⟶ magnetic instability develops with spontanous breaking of TR symmetry.

S.C. Zhang's model — HgTe / CdTe quantum wells.

- conduction band s-orbital
- valance band p-orbital $\longrightarrow$ SO coupling $j_\pm = 1 \pm 1/2 = \begin{cases} 3/2 \\ 1/2 \end{cases}$

$j = 3/2$ band are close to s-one

Angular momentum addition

$$\boxed{Y_{11} \sim \frac{-1}{\sqrt{2}}(P_x + iP_y),\ Y_{10} \sim P_z,\ Y_{1-1} \sim \frac{1}{\sqrt{2}}(P_x - iP_y)}$$

$$Y_{j=3/2,\, j_z=3/2} = \begin{pmatrix} Y_{11} \\ 0 \end{pmatrix} \Rightarrow |3/2\ 3/2\rangle = -\frac{1}{\sqrt{2}}(|P_x\rangle + i|P_y\rangle) \otimes |\uparrow\rangle$$

$$Y_{3/2,\,1/2} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{1+1}\, Y_{10} \\ Y_{11} \end{pmatrix} \Rightarrow |3/2\ 1/2\rangle = \sqrt{\tfrac{2}{3}}\, P_z \otimes |\uparrow\rangle - \sqrt{1/6}\,(|P_x\rangle + i|P_y\rangle) \otimes |\downarrow\rangle$$

$$Y_{3/2\,-1/2} = \frac{1}{\sqrt{3}} \begin{pmatrix} Y_{1-1} \\ \sqrt{2}\, Y_{10} \end{pmatrix} \Rightarrow |3/2\ -1/2\rangle = \sqrt{1/6}\,(|P_x\rangle - i|P_y\rangle) \otimes |\uparrow\rangle + \sqrt{\tfrac{2}{3}}\, |P_z\rangle \otimes |\downarrow\rangle$$

$$Y_{3/2\,-3/2} = \begin{pmatrix} 0 \\ Y_{1-1} \end{pmatrix} \Rightarrow |3/2\ -3/2\rangle = \frac{1}{\sqrt{2}}(P_x - iP_y) \otimes |\downarrow\rangle$$

Time-reversal $\quad T = -i\sigma_y K = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} K$, or $\boxed{\begin{matrix} T|\uparrow\rangle = |\downarrow\rangle \\ T|\downarrow\rangle = -|\uparrow\rangle \end{matrix}}$

$$\Rightarrow T\,[c_1|\uparrow\rangle + c_2|\downarrow\rangle] = T\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} c_1^* \\ c_2^* \end{pmatrix} = \begin{pmatrix} -c_2^* \\ c_1^* \end{pmatrix}$$

integer

$\Rightarrow$ for ~~state~~ $|l m\rangle$, we have $T|lm\rangle = (-)^m\,|l\,-m\rangle \Rightarrow T^2 = 1$

$|j, j_z = m + 1/2\rangle$, we have $T|j, m+1/2\rangle = (-)^m\,|j, -m-1/2\rangle$

$\Downarrow$

$T^2 = -1$

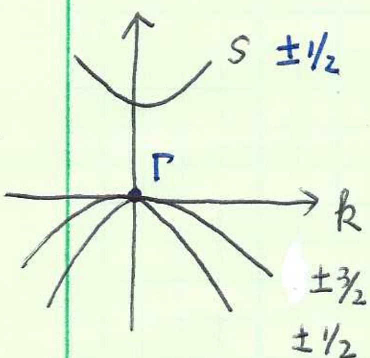
half-integer

① The $j_+ = 3/2$ band, the effective Hamiltoinan around Γ-point — Luttinger Hamiltonian

$$H^P_{3/2} = -\gamma_0 k^2 + \gamma_2 (\vec{k}\cdot\vec{S})^2 \ . \leftarrow \text{ helicity basis}$$

$$\hat{k}\cdot\vec{S}\,|\alpha(k)\rangle = \lambda_\alpha |\alpha(k)\rangle$$

$$H^S_{1/2} = \Delta\mathcal{E} + \gamma_s k^2 \qquad \lambda_\alpha = \pm 3/2, \pm 1/2$$



heavy hole $\hat{k}\cdot\vec{S} = \pm 3/2$

$$|m_{HH}| = 2(\gamma_0 - \tfrac{9}{4}\gamma_2)$$

$$|m_{LH}| = 2(\gamma_0 - \tfrac{1}{4}\gamma_2) > |m_{HH}|$$

(time-reversal, 3D rotation, parity symmetry)

② Now reduce the gap between S and p-band, i.e. set $\Delta\mathcal{E} \to 0$. We need to consider the hybridizatin between S and $P(j=3/2)$ bands. Consider the $\vec{k} \parallel \hat{z}$, then j_z remains conserved along this axis. As a result, the states with heavy hole $|3/2, \pm 3/2\rangle$ do not hybridze with the light hole and electron states. Conside the hybridizatin matrix

$$\begin{array}{ccc} |k_z, S\ \tfrac12\rangle, & |k_z, LH\ \tfrac12\rangle & \\ \left(\begin{array}{c} \Delta\mathcal{E}+\gamma_s k_z^2 \\ v k_z \end{array}\right. & \left.\begin{array}{c} v k_z \\ -\underbrace{(\gamma_0-\tfrac94\gamma_2)}_{\Downarrow\ \gamma_{HH}} k_z^2 \end{array}\right) & \begin{array}{l} |k_z S\ \tfrac12\rangle \\ |k_z LH\ \tfrac12\rangle \end{array} \end{array}$$

Let us use symmetry principle to guild the analysis of matrix element

$$H = M_{12}(k_z)\, C^\dagger_{S\uparrow}(k_z)\, C_{LH,\uparrow}(k_z) + \cdots$$

① under rotation around k_z, $C^\dagger_{S\uparrow}(k_z)\, C_{LH\uparrow}(k_z)$ invariant

$M_{12}(k_z)$ is also invariant.

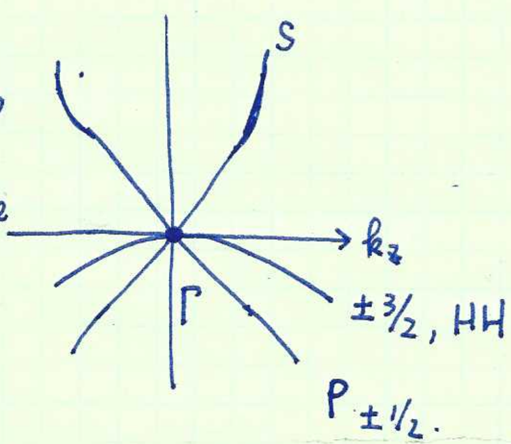
② inversion $\begin{cases} C^\dagger_{S\uparrow}(k_z) \to C^\dagger_{S\uparrow}(-k_z), \quad C_{LH\uparrow}(k_z) \to -C_{LH\uparrow}(-k_z) \\ k_z \to -k_z \end{cases}$

H invariant $\Rightarrow M_{12}(k_z) = -M_{12}(k_z)$

Expand $M_{12}(k_z) = v k_z$.

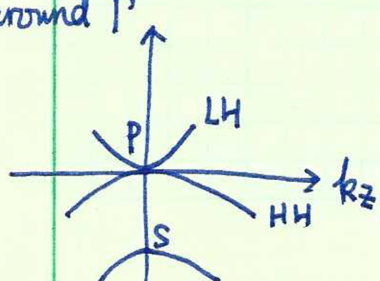
$$\Rightarrow H(k_z) = \left(\frac{\Delta\varepsilon}{2} + \frac{\gamma_S - \gamma_{HH}}{2} k_z^2\right) + \frac{\Delta\varepsilon + (\gamma_S - \gamma_{HH})k_z^2}{2}\tau_z + v k_z \tau_x$$

at $\Delta\varepsilon = 0 \Rightarrow$ triple degenerate point



S↑ and LH 1/2 heavily hybridize. at Γ-point

at $\Delta\varepsilon < 0$, band invert around Γ



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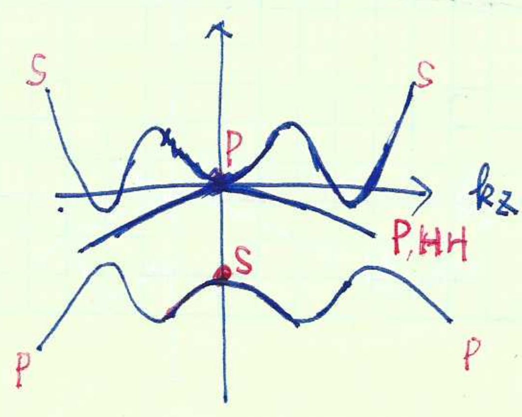
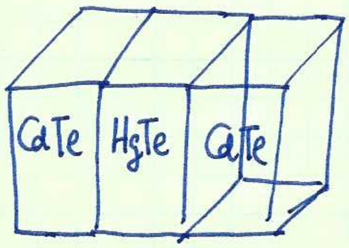


Because of rotation symmetry, the dispersion along other direction $\vec{k}$ is the same. We need to use the helicity basis, i.e. eigen state of spin / angular momentum projection along $\hat{k}$.

3 Put HgTe into 2D Quantum well. k_z is no longer a good Quantum number.



We will use j_z or S_z eigenbasis.

Then HH and LH at $\vec{k}=0$ are no longer degenerate. LH is pushed to even higher energy. We only need to consider the hybridzation of HH and electrons.

$$
\begin{array}{c}
\begin{array}{cccc} |HH\ \tfrac{3}{2}\rangle & |S\ \tfrac{1}{2}\rangle & HH^{-3/2} & |S -\tfrac{1}{2}\rangle \end{array} \\
\begin{array}{cc}
\left(\begin{array}{c|c}
\boxed{\begin{matrix} -\gamma_{HH}k^2 & M_{12} \\ M_{21}^{*} & \Delta\mathcal{E}_s+\gamma_s k^2 \end{matrix}} & \text{negligible} \\ \hline
 & H_{\downarrow}
\end{array}\right)
&
\begin{array}{l} |HH\ |\tfrac{3}{2}\rangle \\ |S\ \tfrac{1}{2}\rangle \\ |HH, -\tfrac{3}{2}\rangle \\ |S -\tfrac{1}{2}\rangle \end{array}
\end{array}
\end{array}
$$

Consider Left-up block: $\rightarrow M_{12}(k_x, k_y)\, C^{+}_{HH,3/2}(k_x,k_y)\, C_{S\,1/2}(k_x,k_y)$

Rotation around z-axis at angle θ

Define rotation $\mathcal{U} = e^{-iJ_z\theta}$

$$\Rightarrow \mathcal{U}^{-1} C^{\dagger}_{HH,3/2}(k_x,k_y)\,\mathcal{U} = C^{\dagger}_{HH,3/2}(Rk_x,Rk_y)\, e^{i\frac{3}{2}\theta}$$

$$\mathcal{U}^{-1} C_{HH,1/2}(k_x,k_y)\,\mathcal{U} = C_{HH,1/2}(Rk_x,Rk_y)\, e^{-i\frac{\theta}{2}}$$

$$\Rightarrow \mathcal{U}^{-1} H\, \mathcal{U} \to M_{12}(k_x,k_y)\, e^{i\theta}\, C^{\dagger}_{HH,3/2}(Rk_x,Rk_y)\, C_{S,1/2}(Rk_x,Rk_y)$$

$$\Rightarrow M(Rk_x, Rk_y) = e^{i\theta} M(k_x,k_y) \Rightarrow M(k_x,k_y) = v(k_x+ik_y)$$

★ Consider the lattice effect: the upp-left block (square lattice)

$$H_\uparrow = \begin{pmatrix} +t_{HH}(\cos k_x+\cos k_y) & v(\sin k_x + i\sin k_y) \\ v(\sin k_x - i\sin k_y) & \Delta\varepsilon_s - t_s(\cos k_x+\cos k_y) \end{pmatrix}$$

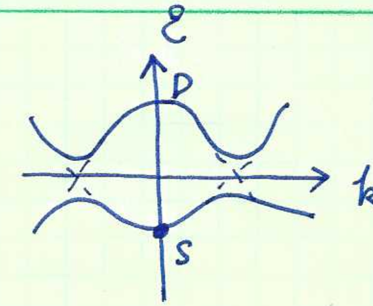
Due to TR symmetry: (Kramers)

HW: Prove that

$$H_\downarrow = \begin{pmatrix} t_{HH}(\cos k_x+\cos k_y) & v(\sin k_x - i\sin k_y) \\ v(\sin k_x + i\sin k_y) & \Delta\varepsilon_s - t_s(\cos k_x+\cos k_y) \end{pmatrix}$$

Hint: $\langle T\psi | T\phi\rangle = \langle \phi|\psi\rangle^*$

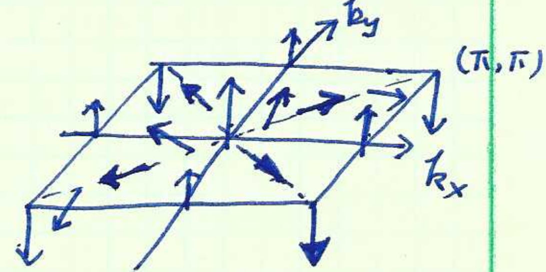
Let us look at
$$H_\uparrow = \left[-\frac{\Delta\varepsilon_s}{2} + \frac{t_{HH}+t_s}{2}(\cos k_x+\cos k_y)\right]\tau_z + v(\sin k_x\,\tau_x + \sin k_y\,\tau_y)$$



$$\begin{cases} d_x = \sin k_x \\ d_y = \sin k_y \\ d_z = \frac{t_{HH}+t_s}{2}(\cos k_x + \cos k_y) - \frac{\Delta \mathcal{E}_s}{2} \end{cases} , \quad \Delta \mathcal{E}_s < 0$$

at $k=(0,0)$ $\quad d_z = t_{HH}+t_s - \frac{\Delta \mathcal{E}_s}{2} > 0$

(π,π) $\quad -(t_H+t_s) - \frac{\Delta \mathcal{E}_s}{2} < 0$



then $\displaystyle\int \frac{dk_x dk_y (\partial_{k_x}\vec{d} \times \partial_{k_y}\vec{d}) \cdot \vec{d}}{4\pi} = \pm 1$

The winding number for $H_\downarrow$ sector is opposite.