Lecture 6 path integral to superfluid.

For a free boson system $H_0 = \sum_k (\epsilon_k - \mu) a_k^\dagger a_k \longrightarrow$

$$\int D^2[a_k(t)] \; e^{i \int dt \, \sum_k [i \frac{1}{2} (a_k^* \dot{a}_k - a_k \dot{a}_k^*) - (\epsilon_k - \mu) a_k^* a_k]}$$

in real-space, we have

$$\int D^2 a \; e^{i \int d^d x \, dt \, [i \frac{1}{2} (\varphi^*(xt) \partial_t \varphi(xt) - \varphi(xt) \partial_t \varphi^*(xt)] - \frac{1}{2m} \partial_x \varphi^*(xt) \partial_x \varphi(xt) - \mu \varphi^*(xt) \varphi(xt)]}, \quad \varphi(xt) = \sum_k e^{ikr} a(r,t)$$

we can also add term of interaction in the action $\int d^d x \, d^d y \, \frac{1}{2} \varphi^*(xt) \varphi^*(yt) V(x-y) \varphi(yt) \varphi(xt)$

By using the short-range inter-action approximation,

$$\mathcal{Z} = \int D\varphi^*(t) D\varphi(t) \; e^{iS}, \quad \text{where}$$

$$S = \int d^d x \, dt \; [i \frac{1}{2} (\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) - \frac{1}{2m} \partial_x \varphi^* \partial_x \varphi + \mu |\varphi|^2 - \frac{V_0}{2} |\varphi|^4]$$

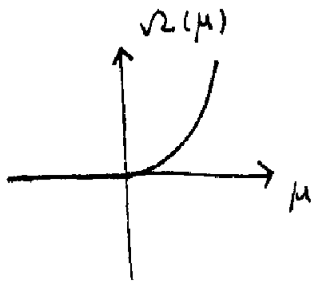
The leading order approximation of thermal potential

$$\Omega = \int d^d x \; \frac{1}{2m} \partial_x \varphi^* \partial_x \varphi - \mu |\varphi|^2 + \frac{V_0}{2} |\varphi|^4$$

§2 symmetry spontanous breaking

a uniform solution $\quad \dfrac{\Omega_0}{V} = -\mu|\varphi_0|^2 + \dfrac{V_0}{2}|\varphi_0|^4$, minimizing Ω_0

$$\Rightarrow \quad \varphi_0 = \begin{cases} \sqrt{\dfrac{\mu}{V_0}}\, e^{i\theta} & \text{for } \mu > 0 \\ 0 & \text{for } \mu \le 0 \end{cases} \quad \Rightarrow \Omega_0(\mu) = \begin{cases} \dfrac{\mu^2}{V_0} & \mu > 0 \\ 0 & \mu < 0. \end{cases}$$



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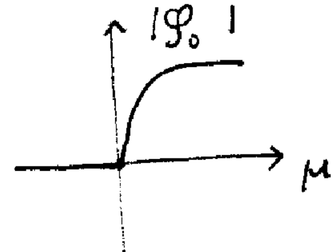
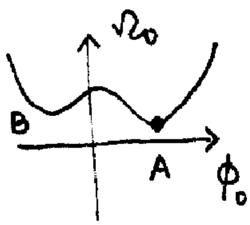


$U(1)$- symmetry : $\quad \varphi \to e^{i\theta}\varphi$, $\quad S$ does not change.

However, the classical ground state has a fixed value of θ, which breaks the $U(1)$ symmetry. A phase transition is related to a singlurarity of the thermo-dynamic potential. v.s. μ.



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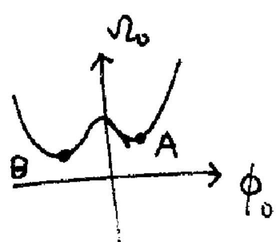
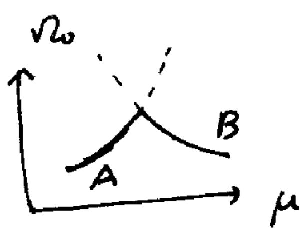
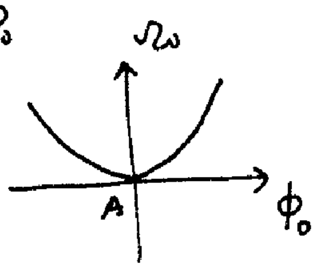




First order

$\phi_0 \to -\phi_0$ symmetry



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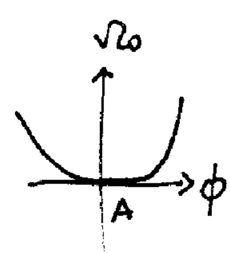

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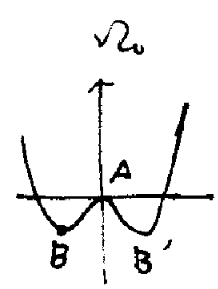


→ second order phase transition

breaking symmetry

Long-range order $\langle \varphi^\dagger(x,t)\, \varphi(0,0) \rangle \Big|_{(x,t)\to\infty} = |\varphi_0|^2 \neq 0$

The symmetry breaking phase can be either represented by a nonzero order parameter, or a long range correlation.

§ 3. low energy effective theory

$\varphi = \varphi_0 + \delta\varphi$ at $\mu < 0 \Rightarrow \varphi_0$

$\Rightarrow$ $S = \int d^dx\, dt \, [i \frac{1}{2} (\delta\varphi^* \partial_t \delta\varphi - c.c.) - \frac{1}{2m} \partial_x \delta\varphi^* \partial_x \delta\varphi + \mu|\varphi|^2]$

$\rightarrow$ equation of motion: $[i\partial_t \delta\varphi - \frac{1}{2m} (-i\partial_x)^2 + \mu]\ \varphi = 0$

$\omega = \frac{1}{2m} k^2 - \mu$, free boson.

At the symmetry-breaking ground state, we parameterize the the ground state field configuration as $\varphi = \sqrt{\rho_0 + \delta\rho}\ e^{i\theta}$, $(\rho_0 = \mu/V_0)$

$$\Rightarrow \quad S = \int d^dx\, dt\, [-(\rho_0 + \delta\rho)\partial_t\theta - \frac{\rho_0(\partial_x\theta)^2}{2m} - \frac{(\partial_x\delta\rho)^2}{8m\rho_0} - \frac{V_0}{2}(\delta\rho)^2$$

(keep to quadratic order of $\delta\rho$ and θ).

θ describes the slow, low-frequency mode, $\delta\rho$ describes the massive fluctuations if we set $\delta\rho = 0 \Rightarrow \quad S = \int d^d x\, dt \; -\rho_0 \partial_t \theta - \frac{1}{2m} \rho_0 (\partial_x \theta)^2$

which is not correct, because $\partial_t \theta$ is a total derivative and does not enter equation of motion. We need to integrate out the massive field $\delta\rho$

$$\mathcal{Z} = \int D\theta \, D\delta\rho \; e^{i\int d^d x dt \left[\frac{-V_0}{2} (\delta\rho)^2 - \frac{(\partial_x \delta\rho)^2}{8m\rho_0} \right] - \delta\rho\, \partial_t \theta + \frac{1}{2m} \delta\rho (\partial_x^2 \theta)}$$

integral over $\delta\rho$

$$\hookrightarrow \int D\delta\rho \; e^{i\int d^d x\, dt \; -\frac{1}{2} \left(\delta\rho + \frac{\partial_t \theta}{V_0 - \frac{\partial_x^2}{4m\rho_0}}\right) \left[V_0 - \frac{\partial_x^2}{4m\rho_0} \right] \left(\delta\rho + \frac{\partial_t \theta \cdots}{V_0 - \frac{\partial_x^2}{4m\rho_0}}\right)}$$

$$\cdot \, e^{i\int d^d x dt \; \frac{1}{2} \left[\partial_t \theta \cdot \qquad \right] \frac{1}{V_0 - \frac{1}{4m\rho_0} \partial_x^2} \left[\partial_t \theta \qquad \right]}$$

$$\approx \text{Const} \cdot e^{i\int d^d x dt \; \frac{1}{2} [\partial_t \theta] \frac{1}{V_0} (\partial_t \theta)}$$

$$\Rightarrow \boxed{\mathcal{Z} = \int D\theta \; e^{i\int d^d x\, dt \; \frac{1}{2V_0} (\partial_t \theta)^2 - \frac{\rho_0}{2m} (\partial_x \theta)^2}}$$

θ lives on a circle, θ and $\theta + 2\pi$ refer to the same point.

we can introduce $z = e^{i\theta} \Rightarrow S_{eff} = \int d^d x dt \left\{ \frac{1}{2V_0} |\partial_t z|^2 - \frac{\rho_0}{2m} |\partial_x z|^2 \right\}$

X-y model

equation of motion: $\Theta(x,t) = \bar{\theta} + \delta\theta \Rightarrow$

$$\left(-\frac{1}{2V_0}\partial_t^2 + \frac{\rho_0}{2m}\partial_x^2\right)\delta\theta = 0 \Rightarrow \omega = v|k| \text{ with } v^2 = \frac{\rho_0 V_0}{m}.$$

Density & current:

we add source term $-A_0\rho = -A_0(\rho_0 + \delta\rho)$, $\vec{A}\cdot\vec{j} = \vec{A}\cdot \mathrm{Re}(\varphi^\dagger \frac{\partial_x}{im}\varphi)$

$$= \vec{A}\,\frac{\rho_0}{m}\nabla_x\theta$$

$$\int D\delta\rho\; e^{i\int d^dx dt\, -\frac{1}{2}\left[\delta\rho + \frac{\partial_t\theta + A_0}{V_0}\right]^2 \frac{1}{V_0}} \cdot \exp\left[i\int d^dx dt\, \frac{1}{2V_0}(\partial_t\theta + A_0)^2\right.$$

$$\rightarrow \quad \frac{1}{V_0}\partial_t\theta\cdot A_0 - A_0\rho_0 \Rightarrow \boxed{\rho = \rho_0 - \frac{\partial_t\theta}{V_0}} \qquad \vec{j} = \frac{\rho_0}{m}\nabla_x\theta$$

§ phonon mode

In the above discusse θ is ansidered as waves. According to the particle-wave duality, we can also consider them as bosons.

$$\mathcal{L}_{eff} = \sum_k \frac{A_k}{2}\dot{\theta}_{-k}\dot{\theta}_k - \frac{B_k}{2}\theta_{-k}\theta_k, \quad A_k = V_0^{-1},\ B_k = \frac{\rho_0 k^2}{m}$$

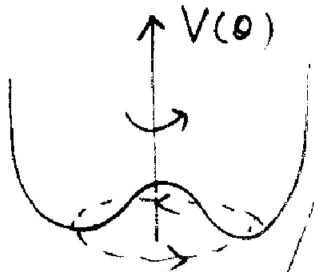
$$\pi_k = \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\theta}_k} = A_k\dot{\theta}_{-k} \rightarrow H = \sum_k\left(\frac{1}{2A_k}\pi_{-k}\pi_k + \frac{B_k}{2}\theta_{-k}\theta_k\right)$$

introduce $\alpha_k = u_k\theta_k + i v_k \pi_{-k}$, $u_k = \frac{1}{\sqrt{2}}(A_kB_k)^{1/4}$, $v_k = \frac{1}{\sqrt{2}}(A_kB_k)^{-1/4}$

we get $H = \sum_k \varepsilon_k \alpha_k^\dagger \alpha_k$

§ Goldstone modes & spontaneous symmetry breaking

U(1) symmetry breaking $\varphi \to \varphi e^{i\theta}$, but the ground state does not have such a symmetry. Nambu, Goldstone



proved that there's a gapless mode, which describe that the system is trying to restore the symmetry, i.e. the fluctuations among degenerate ground states.

Different degenerate ground states are related by symmetry transformation.

One can write low energy theory in terms of Goldstone modes, which is usually called non-linear-σ model.

• Spontaneous symmetry can only happen in infinite systems. At finite systems, quantum fluctuations will restore the full symmetry. But for infinite system, it takes an infinitely long time to fluctuate from one state with one value of order parameter to another state by a different order parameter.

For example, for the superfluid boson state $\varphi_0 = \langle \Omega | a | \Omega \rangle = |\varphi_0| e^{i\theta}$.

Let us define $W = e^{i\hat{N}\theta}$, we know $[W, H] = 0$.

$$W\, a(x)\, W^{\dagger} = a(x)\, e^{i\theta},$$ $\Rightarrow$ For finite system, $|\Omega\rangle$ is $\hat{N}$ eigenstate

$$\langle\Omega|\, a\, |\Omega\rangle = \langle\Omega|\, W\, a(x)\, W^{\dagger}\, |\Omega\rangle = e^{i\theta}\, \langle\Omega|\, a\, |\Omega\rangle = 0.$$

Let us look at as the volume $\to +\infty$, how the fluctuation slow down and lead to a symmetry breaking ground state. We only consider uniform fluctuations

$$\mathcal{L} = V \cdot \left[\frac{1}{2V_0} (\partial_t \theta)^2 - \rho_0 \partial_t \theta + \mu \rho_0 - \frac{1}{2} V_0 \rho_0^2 \right],$$

which describe a particle moving on a circle with $\theta \in (0, 2\pi)$. The mass of the particle $M = V/V_0$, and $p = \partial_{\dot\theta} \mathcal{L} = M\dot\theta - \frac{\mu V}{V_0} = -N$.

$\Rightarrow$ $H = p\dot\theta - \mathcal{L} = \dfrac{p^2}{2M} + \dfrac{\mu V}{V_0} p$. As $V \to +\infty$, the particle becomes classic, and does not move i.e. $\dot\theta = 0$.

But as V is finite, p is quantized with momentum $-n$ $\Rightarrow$

$$E_n = \frac{n^2}{2M} - \frac{\mu V n}{V_0 M} = \frac{V_0 n^2}{2V} - \mu n$$

The state with $p = -n = \dfrac{-\mu V}{V_0}$ has the minimum energy, which can be written as

$$|\Phi_0\rangle = \int d\theta\; e^{-i \frac{\mu V}{V_0} \theta_0}\; |\theta_0\rangle .$$

$\hat{N}$ is the conjugate operator to θ, $\hat{N} = -P_\theta = +i\hbar \frac{\partial}{\partial\theta}$. Thus $\hat{N}$-eigenstate cannot have a well defined θ-value.

But as $V \to +\infty$, the low lying state with different $\hat{N}$, are nearly degenerate

$$\underbrace{|\theta\rangle = \sum_n e^{in\theta} |n\rangle}_{\text{the symmetry breaking state.}}, \quad \text{the gap vanishes as } \frac{1}{V}.$$