

Lecture 11. Landau Fermi liquid (IV)

§ Ward identities — Abrikosov et al. P158

Consider an external small perturbation $H'_{(t)} = \int \delta u(t) \psi_{\alpha}^{\dagger}(r) \psi_{\alpha}(r) dr$.

We study the variation of Green's function:

$$i G_{\alpha\beta}(x; x') = \frac{\int D\bar{\psi} D\psi \psi_{\alpha}(x) \bar{\psi}_{\beta}(x') e^{i\int dt (L+L')}}{\int D\bar{\psi} D\psi e^{i\int dt L+L'}}$$

where $L' = -H'(t)$. correct to the linear order of $\delta u(t)$, we have

$$i G_{\alpha\beta}(x, x') = \frac{\int D\bar{\psi} D\psi \psi_{\alpha}(x) \bar{\psi}_{\beta}(x') e^{i\int dt L} [1 + i\int dt L']}{\int D\bar{\psi} D\psi e^{i\int dt L} [1 + i\int dt L']} / \frac{\int D\bar{\psi} D\psi e^{i\int dt L}}{\int D\bar{\psi} D\psi e^{i\int dt L}}$$

$$= \frac{\int D\bar{\psi} D\psi \psi_{\alpha}(x) \bar{\psi}_{\beta}(x') e^{i\int dt L}}{\int D\bar{\psi} D\psi e^{i\int dt L}} / \frac{\int D\bar{\psi} D\psi e^{i\int dt L}}{\int D\bar{\psi} D\psi e^{i\int dt L}}$$

$$+ \frac{\int D\bar{\psi} D\psi [\psi_{\alpha}(x) \bar{\psi}_{\beta}(x') i\int dt L'] e^{i\int dt L}}{\int D\bar{\psi} D\psi e^{i\int dt L}} / \frac{\int D\bar{\psi} D\psi e^{i\int dt L}}{\int D\bar{\psi} D\psi e^{i\int dt L}}$$

$$\ominus \frac{\int D\bar{\psi} D\psi (\psi_{\alpha}(x) \bar{\psi}_{\beta}(x') e^{i\int dt L'})}{\int D\bar{\psi} D\psi e^{i\int dt L}} \cdot \frac{\int D\bar{\psi} D\psi (i\int dt L') e^{i\int dt L}}{\int D\bar{\psi} D\psi e^{i\int dt L}}$$

$$\rightarrow i \delta G_{\alpha\beta}(x, x') = \langle \psi_{\alpha}(x) \bar{\psi}_{\beta}(x') (-i) \int dy \delta u(t_y) \bar{\psi}_{\delta}(y) \psi_{\delta}(y) \rangle$$

$$- \langle \psi_{\alpha}(x) \bar{\psi}_{\beta}(x') \rangle \langle (-i) \int dy \delta u(t_y) \bar{\psi}_{\delta}(y) \psi_{\delta}(y) \rangle$$

← Path integral average, or time-ordered in operator language

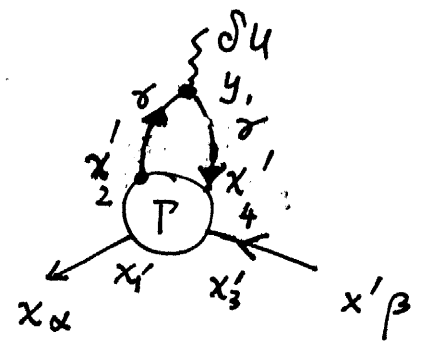
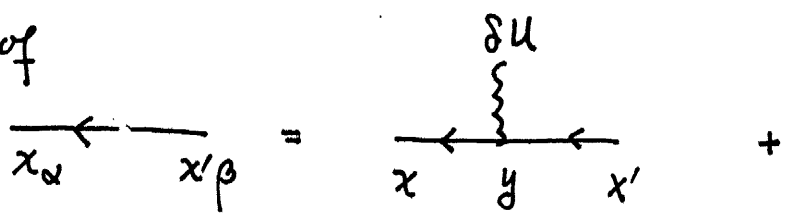
According to the definition of the vertex function

$$\begin{aligned}
 - \langle \psi(x_1) \psi(x_2) \psi^\dagger(x_3) \psi^\dagger(x_4) \rangle &= - \langle \psi(x_1) \psi^\dagger(x_3) \rangle \langle \psi(x_2) \psi^\dagger(x_4) \rangle + \langle \psi(x_1) \psi^\dagger(x_4) \rangle \langle \psi(x_2) \psi^\dagger(x_3) \rangle \\
 + i \int d^4x'_1 d^4x'_2 d^4x'_3 d^4x'_4 &\langle \psi(x_1) \psi^\dagger(x'_1) \rangle \langle \psi(x_2) \psi^\dagger(x'_2) \rangle \langle \psi(x'_3) \psi^\dagger(x_3) \rangle \langle \psi(x'_4) \psi^\dagger(x_4) \rangle \\
 &\Gamma(x'_1, x'_2; x'_3, x'_4) \quad \leftarrow \begin{matrix} x_1 = x & x_3 = x' \\ x_2 = y & x_4 = y \end{matrix}
 \end{aligned}$$

$$\Rightarrow \delta G_{\alpha\beta}(x, x') = \delta_{\alpha\beta} \int d^4y \delta U(t_y) G(x-y) G(y-x')$$

$$\begin{aligned}
 - i \int d^4y d^4x'_1 d^4x'_2 d^4x'_3 d^4x'_4 &\delta U(t_y) G(x-x'_1) G(y-x'_2) G(x'_3-x') \\
 &G(x'_4-y) \\
 &\Gamma_{\alpha\beta; \gamma\delta}(x'_1, x'_2; x'_3, x'_4)
 \end{aligned}$$

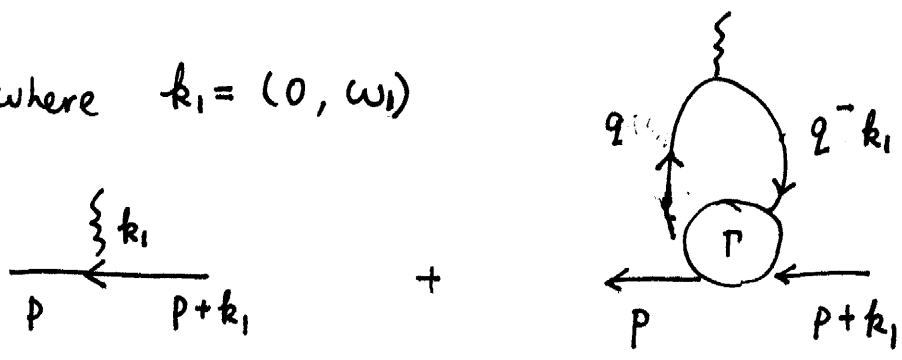
change of



→ Fourier

$$\begin{aligned}
 \delta G_{\alpha\beta}(p; p+k_1) &= \delta_{\alpha\beta} G(p) \delta U(\omega) G(p+k_1) - i G(p) G(p+k_1) \\
 &\times \int \Gamma_{\alpha\beta; \gamma\delta}(p, q; k_1) G(q) \delta U(\omega) G(q-k_1) \frac{d^4q}{(2\pi)^4}
 \end{aligned}$$

where $k_1 = (0, \omega)$



taking 1/2 of the trace

$$- \delta G = G(p) \delta u(\omega_1) G(p+k_1) - i G(p) G(p+k_1) \frac{1}{2} \int \Gamma_{\alpha\beta;\alpha\beta}(p, q; k_1)$$

$$\otimes G(q) \delta u(\omega_1) G(q-k_1) \frac{d^4 q}{(2\pi)^4}$$

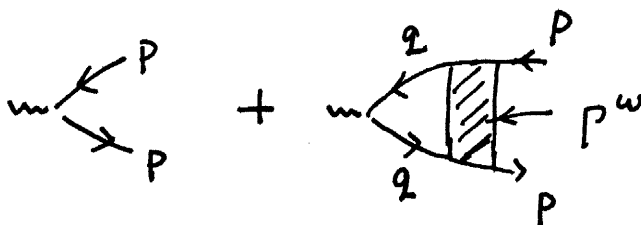
Set $\omega_1 \rightarrow 0$, which means we add $-\delta u$ into the frequency of

$$G(p) \Rightarrow \frac{\delta G}{\delta u} = - \frac{\partial G}{\partial \omega}$$

$$\Rightarrow \frac{\partial G}{\partial \omega} = - [G^2(p)]_\omega \left[1 - \frac{i}{2} \int \Gamma_{\alpha\beta;\alpha\beta}^\omega(p, q) [G^2(q)]_\omega \frac{d^4 q}{(2\pi)^4} \right]$$

we are considering the limit of $\lim_{\omega \rightarrow 0} \lim_{k \rightarrow 0}$

$$\Rightarrow \frac{\partial G^{-1}(p)}{\partial \omega} = \frac{1}{\omega} = 1 - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^\omega(p, q) [G^2(q)]_\omega$$



② Now let's consider the particle has charge

$$\frac{(\vec{p} - \frac{e}{c} \vec{A})^2}{2m}$$

$$\rightarrow H' = - \frac{e}{mc} \int \psi_\alpha^\dagger(r) \hat{p} \psi_\alpha(r) d^3 r. \text{ Let's consider the static}$$

but slowly varying limit $\lim_{k \rightarrow 0} \lim_{\omega \rightarrow 0} \leftarrow$ Static small B field.

⇒ Similarly, we get the variation of

$$\delta G = -G(p) \frac{e}{mc} (\vec{p} \cdot \vec{A}) G(p+k_2) + \frac{i}{2} G(p) G(p+k_2) \times \int \frac{d^4 q}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}(pq; k_2) \frac{e}{mc} (\vec{q} \cdot \vec{A}) G(q) G(q+k_2)$$

where $k_2 = (\vec{k}, 0)$. Set \vec{A} to be small, this means

$$p \rightarrow p - \frac{e}{c} A$$

$$\Rightarrow \delta G = \frac{\partial G}{\partial \vec{p}} \left(-\frac{e}{c} \vec{A}\right) \Rightarrow$$

$$-\frac{\partial G}{\partial \vec{p}} = -\left(G^2(p)\right)_k \frac{\vec{p}}{m} + \frac{i}{2} \left\{G^2(p)\right\}_k \int \frac{d^4 q}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}^k(pq) G^2(q) \frac{\vec{q}}{m}$$

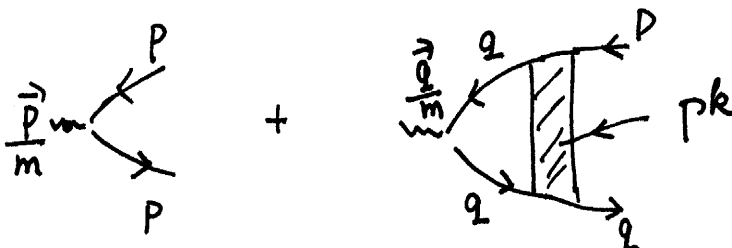
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$$\Rightarrow \frac{\partial G^2(p)}{\partial \vec{p}} = -\frac{\vec{p}}{m} + \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}^k(pq) \left[G^2(q)\right]_k \frac{\vec{q}}{m}$$

↑

$$\frac{-\vec{v}}{Z} = -\frac{\vec{p}}{m^* Z}, \quad \text{i.e.}$$

$$\frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{m} - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}^k(pq) \left[G^2(q)\right]_k \frac{\vec{q}}{m}$$



Current vertices

③ Let us consider a boost $H' = -\vec{v} \cdot \vec{p} = -\vec{v} \cdot \int \psi_{\alpha}^{\dagger}(\mathbf{r}) (-i\hbar \nabla) \psi_{\alpha}(\mathbf{r}) d\mathbf{r}$ ⑤

\Rightarrow the variation of Green's function. We consider the $\lim_{\omega \rightarrow 0} \lim_{\mathbf{k} \rightarrow 0}$

$$\delta G = -G(p) (\vec{p} \cdot \vec{v}) G(p+k_1) \rightarrow \text{small E-field}$$

$$+ \frac{i}{2} G(p) G(p+k_1) \int \Gamma_{\alpha\beta; \alpha\beta}(p, q; k_1) (\vec{q} \cdot \vec{v}) G(q) G(q-k_2) \frac{d^4 q}{(2\pi)^4}$$

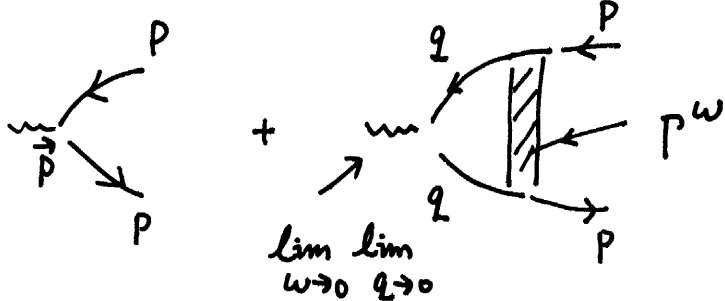
where $k_1 = (0, \omega_1)$. The doppler shift the spectra $E_p \rightarrow E_p - \vec{p} \cdot \vec{v}$,

which is equivalent to $\omega \rightarrow \omega + \vec{p} \cdot \vec{v}$. Set $\lim_{\omega_1 \rightarrow 0} \lim_{\mathbf{k}_1 \rightarrow 0}$

$$\delta G = \frac{\partial G}{\partial \omega} \vec{p} \cdot \vec{v} \Rightarrow -G^2(p) \frac{\partial G}{\partial \omega} \vec{p} = \vec{p} - \frac{i}{2} \int \Gamma_{\alpha\beta}^{\omega}(p, q) \vec{q} [G(q)]_{\omega}^2$$

$$\Rightarrow \frac{\partial G^{-1}}{\partial \omega} \vec{p} = \vec{p} - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\beta}^{\omega}(p, q) \vec{q} [G(q)]_{\omega}^2 = \frac{\vec{p}}{z}$$

$$\downarrow \frac{\vec{p}}{z}$$



④ Let us consider a variation of $U(r)$ in the limit of $\lim_{\mathbf{k} \rightarrow 0} \lim_{\omega \rightarrow 0}$ (static limit). $\Rightarrow H' = \int d\mathbf{r} \delta U(\mathbf{r}) \psi^{\dagger}(\mathbf{r}) \psi(\mathbf{r})$

$$\delta G = G(p) \delta U(\vec{k}) G(p+k_2) - \frac{i}{2} G(p) G(p+k_2) \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\beta; \alpha\beta}(p, q; k_2) G(q) \delta U(\vec{k}) G(q+k_2)$$

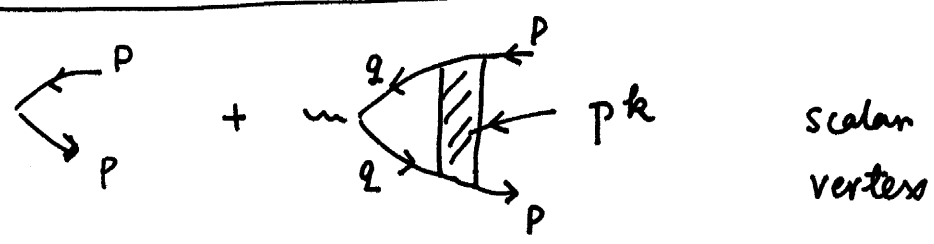
where $k_2 = (\vec{k}, 0)$

In the limit of $k \rightarrow 0$, this corresponds to $\mu + \delta U = \text{const}$

thus $\delta\mu = -\delta U \Rightarrow \delta G = \frac{\partial G}{\partial \mu} (-\delta U)$

$\Rightarrow -\frac{\partial G}{\partial \mu} = G^2(p) + \frac{i}{2} G^2(p) \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^k(p, q) [G^2(q)]_k$

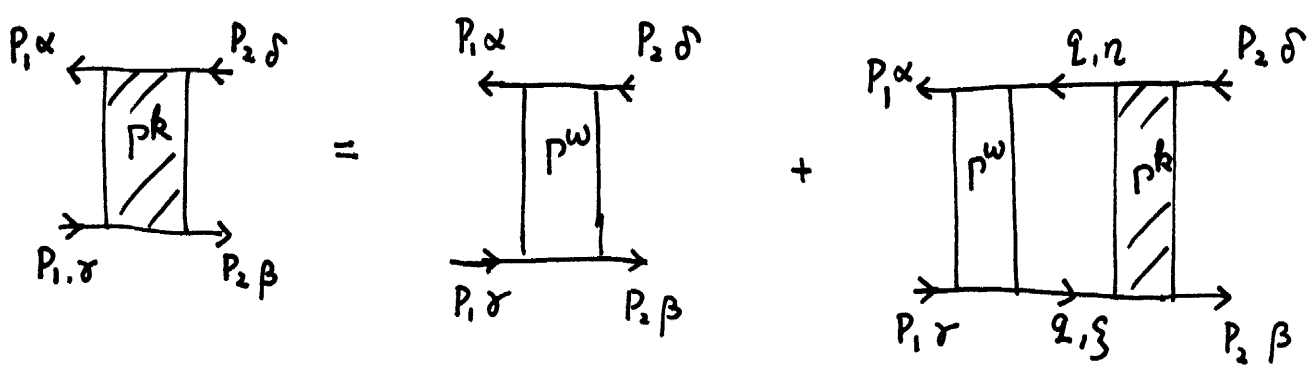
$\Rightarrow \frac{\partial G^{-1}}{\partial \mu} = 1 - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^k(p, q) [G^2(q)]_k$



§ Fermi surface volume

Correct close to Fermi Surface

We need to use $\Gamma_{\alpha\beta;\alpha\delta}^k(p, p_2) = \Gamma_{\alpha\beta;\alpha\delta}^w(p, p_2) - \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int \Gamma_{\alpha\gamma;\alpha\eta}^w(p, q) \Gamma_{\eta\beta;\delta\delta}^k(q, p_2)$



plug in this Eq into

$\frac{\vec{p}}{m^* z} = \frac{\vec{p}}{m} - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^k(p, q) [G^2(q)]_k \frac{\vec{q}}{m}$

$$\Rightarrow \frac{\vec{p}}{m^*z} - \frac{\vec{p}}{m} = -\frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \frac{\vec{q}}{m} [G^2(q)]_k$$

$$+ \frac{1}{2} \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \left(-\frac{\vec{q}}{m^*z} + \frac{\vec{q}}{m} \right)$$

by using the previous Eq again.

plug in $[G^2(p)]_k = [G^2(p)]_\omega - \frac{2\pi i z^2}{v_F} \delta(\omega) \delta(|p| - k_F)$

$$\Rightarrow \frac{\vec{p}}{m^*z} = \frac{\vec{p}}{m} - \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \frac{\vec{q}}{m} [G^2(p)]_\omega$$

$$- \frac{\pi z^2}{v_F} \int \frac{d^4q}{(2\pi)^4} \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \delta(\omega) \delta(|q| - k_F) \frac{\vec{q}}{m}$$

$$+ \frac{1}{2} \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \left(\frac{-\vec{q}}{m^*z} + \frac{\vec{q}}{m} \right)$$

The first line = $\frac{\vec{p}}{zm}$

The second line = $-\frac{\pi z^2}{v_F} \frac{1}{(2\pi)^4} \cdot k_F^2 \int d\Omega \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \frac{\vec{q}}{m}$

$$\Rightarrow \frac{\vec{p}}{m^*z} = \frac{\vec{p}}{zm} - \frac{1}{2} \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \frac{\vec{q}}{m^*}$$

\Rightarrow

$$\frac{1}{m^*} = \frac{1}{m} - \frac{k_F^2 z^2}{2(2\pi)^3} \int d\Omega \frac{z^2}{v_F} \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) \frac{\vec{q}}{k_F}$$

$$\frac{z^2}{2} \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) = 2 f^S(p, q)$$

$$\Rightarrow \frac{1}{m^*} = \frac{1}{m} - \frac{\tilde{N}_0}{m} \frac{1}{3} f_1^S$$

where $f_1^S = \frac{1}{2} \int_{-1}^1 d\omega \sin\theta f^S(\omega \sin\theta) \cos\theta$

and $\tilde{N}_0 = \frac{2 \cdot k_f^2 \cdot 4\pi}{(2\pi)^3 v_f^0} = \frac{2 k_f m \cdot 4\pi}{(2\pi)^3}$

where \tilde{N}_0 is the DOS of free Fermi gas.

$$\text{or } \frac{1}{m^*} = \frac{1}{m} - \frac{N_0}{m^*} \frac{1}{3} f_1^S \Rightarrow \frac{1 + \frac{F_1^S}{3}}{m^*} = \frac{1}{m} \quad \text{or } \boxed{\frac{m^*}{m} = 1 + \frac{F_1^S}{3}}$$

N_0 is the DOS of Fermi liquid.

§ Luttinger theorem:

we prove the Fermi wave vector does not change in Fermi liquid.

Consider $G(p, \omega) = \frac{z}{\omega - v_F(|p| - k_F) + i \text{sgn} \omega \eta} + \dots$

all of z, k_F, v_F depends on μ . the contribution to $\frac{\partial G}{\partial \mu}$ mainly
However,

comes from $\frac{\partial k_F}{\partial \mu}$:

$$\frac{\partial G}{\partial \mu} = \frac{\frac{\partial z}{\partial \mu}}{\omega - v_F(|p| - k_F) + i \text{sgn} \omega \eta} + \frac{-z}{[\omega - v_F(|p| - k_F) + i \text{sgn} \omega \eta]^2} \left[-\frac{\partial v_F}{\partial \mu} (|p| - k_F) + v_F \frac{\partial k_F}{\partial \mu} \right]$$

as $|p| \rightarrow k_F$ and $\omega \rightarrow 0$, the last term dominates

$$\Rightarrow \frac{\partial G}{\partial \mu} \approx -G^2 \frac{v_F}{z} \frac{dk_F}{d\mu} \quad \text{or} \quad \boxed{v_F \frac{dk_F}{d\mu} = z \left(\frac{\partial G^{-1}}{\partial \mu} \right) \Big|_{\substack{\omega=0 \\ |p|=k_F}}}$$

Plug in $\frac{\partial \tilde{G}^{-1}(p)}{\partial \mu} = 1 - \frac{i}{2} \int \Gamma_{\alpha\beta;\alpha\beta}^k(p, q) \{G^2(q)\}_k \frac{d^4 q}{(2\pi)^4}$ (9)

and $\Gamma_{\alpha\beta;\alpha\beta}^k(p, q) = \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) - \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega_{q'} \Gamma_{\alpha\xi;\alpha\xi}^w(p, q') \Gamma_{2\beta;\xi\beta}^k(q', q)$

$$\Rightarrow \frac{v_F}{z} \frac{dk_F}{d\mu} = \left. \frac{\partial \tilde{G}^{-1}(p)}{\partial \mu} \right|_{\omega=0, |p|=k_F} = 1 - \frac{i}{2} \int \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) \{G^2(q)\}_k \frac{d^4 q}{(2\pi)^4} + \frac{i}{2} \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega_{q'} \int \frac{d^4 q}{(2\pi)^4} \Gamma_{\alpha\xi;\alpha\xi}^w(p, q') \Gamma_{2\beta;\xi\beta}^k(q', q) \{G^2(q)\}_k$$

the second line: after integrate $\frac{d^4 q}{(2\pi)^4}$

$$\rightarrow - \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega_{q'} \Gamma_{\alpha\xi;\alpha\xi}^w(p, q') \frac{1}{2} \left[\frac{\partial \tilde{G}^{-1}(q')}{\partial \mu} - 1 \right] \delta_{2\xi}$$

← (η has to be equal to ξ , otherwise, vertices $\Gamma_{\alpha\xi;\alpha\xi}$ and $\Gamma_{2\beta;\xi\beta}$ are not spin conserved)

$$\Rightarrow \frac{v_F}{z} \frac{dk_F}{d\mu} = 1 - \frac{i}{2} \int \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) \{G^2(q)\}_k \frac{d^4 q}{(2\pi)^4} - \frac{1}{2} \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) \left(\frac{v_F}{z} \frac{dk_F}{d\mu} - 1 \right) d\Omega_{q'}$$

using $\{G^2(q)\}_k = \{G^2(q)\}_\omega - \frac{2\pi i z^2}{v_F} \delta(\epsilon) \delta(|q| - k_F)$

The first line: $1 - \frac{i}{2} \int \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) \{G^2(q)\}_\omega \frac{d^4 q}{(2\pi)^4} + \frac{i}{2} \frac{2\pi i z^2}{v_F}$

$$\otimes \int \Gamma_{\alpha\beta;\alpha\beta}^w(p, q) \delta(\omega_q) \delta(|q| - k_F) \frac{d^4 q}{(2\pi)^4} = \frac{1}{z} - \frac{2\pi z^2}{2v_F} \frac{k_F^2}{(2\pi)^4} \int \Gamma_{\alpha\beta}^w(p, q) d\Omega_{q'}$$

$$\frac{v_F}{z} \frac{dk_F}{d\mu} = \frac{1}{z} - \frac{k_F^2 z^2}{2(2\pi)^3 v_F} \frac{v_F}{z} \frac{dk_F}{d\mu} \int \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) d\Omega_q$$

$$\Rightarrow \boxed{v_F \frac{dk_F}{d\mu} = \left(1 + \frac{k_F^2}{2(2\pi)^3 v_F} \int z^2 \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) d\Omega_q\right)^{-1}} = \frac{1}{1+F_0}$$

$$v_F = \frac{dE_F}{dk_F} \Rightarrow \frac{dE_F}{d\mu} = \frac{1}{1+F_0} \quad \text{i.e.} \quad \boxed{\frac{dn}{d\mu} = \frac{dn}{dE} \frac{dE}{d\mu} = \frac{N_0}{1+F_0}}$$

From the definition of Green's function

$$n(k) = \langle \psi^\dagger(k) \psi(k) \rangle = -\langle T [\psi(k,t) \psi^\dagger(k,t+0^+)] \rangle = -i G(k, -0^+) \\ = -i \int \frac{d\omega}{2\pi} G(p, \omega) e^{i\omega\eta} \quad \text{where } \eta = 0^+$$

Taking into account spin \Rightarrow

$$n = \frac{N}{V} = -2i \int \frac{d^3p}{(2\pi)^3} \frac{d\omega}{2\pi} G(p, \omega) e^{i\omega\eta} \quad \leftarrow \eta \rightarrow 0^+$$

$$\frac{dn}{d\mu} = -2i \int \frac{\partial G(p)}{\partial \mu} \frac{d^4p}{(2\pi)^4} = 2i \int \frac{\partial G^{-1}(p)}{\partial \mu} \{G^2(p)\}_k \frac{d^4p}{(2\pi)^4}$$

Plug in $\frac{\partial G^{-1}(p)}{\partial \mu} = 1 - \frac{i}{z} \int \Gamma_{\alpha\beta;\alpha\beta}^k(p,q) \{G^2(q)\}_k \frac{d^4q}{(2\pi)^4}$

$$\Rightarrow \frac{dn}{d\mu} = 2i \int \{G^2(p)\}_k \frac{d^4p}{(2\pi)^4} + \int \Gamma_{\alpha\beta;\alpha\beta}^k(p,q) \{G^2(q)\}_k \frac{d^4q}{(2\pi)^4} \frac{d^4p}{(2\pi)^4}$$

Again, using

$$\Gamma_{\alpha\beta;\alpha\beta}^k(p,q) = \Gamma_{\alpha\beta;\alpha\beta}^w(p,q) - \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega_{q'} \Gamma_{\alpha\gamma;\alpha\gamma}^w(p,q') \Gamma_{\eta\beta;\eta\beta}^k(q',q)$$

the second term goes

$$\int \{G^2(p)\}_k P_{\alpha\beta;\alpha\beta}^{\omega}(p, q) \{G^2(q)\}_k \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} - \frac{k_F^2 z^2}{(2\pi)^3 v_F} \int \frac{d^4 p}{(2\pi)^4} \{G^2(p)\}_k$$

$$\otimes \int d\Omega_{q'} P_{\alpha\beta;\alpha\beta}^{\omega}(p, q') \underbrace{\int \frac{d^4 q}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}^k(q', q) \{G^2(q)\}_k}_{\left[\frac{\partial G^{-1}(q')}{\partial \mu} - 1 \right] 2i \frac{\delta n}{2}}$$

$$\Rightarrow \frac{dn}{d\mu} = 2i \int \{G^2(p)\}_k \frac{d^4 p}{(2\pi)^4} + \int \{G^2(p)\}_k P_{\alpha\beta;\alpha\beta}^{\omega}(p, q) \{G^2(q)\}_k \frac{d^4 q}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} - \frac{i k_F^2 z^2}{(2\pi)^3 v_F} \int \frac{d^4 p}{(2\pi)^4} \underbrace{\{G^2(p)\}_k}_{d\Omega_{q'}} P_{\alpha\beta;\alpha\beta}^{\omega}(p, q') \left[\frac{v}{z} \frac{dk_F}{d\mu} - 1 \right]$$

$$\text{us } \{G^2(p)\}_k = \{G^2(p)\}_\omega - \frac{2\pi i z^2}{v_F} \delta(\omega) \delta(|p| - k_F)$$

The first term \Rightarrow

$$2i \int \{G^2(p)\}_\omega \frac{d^4 p}{(2\pi)^4} + \frac{4\pi z^2}{v_F} \frac{4\pi k_F^2}{(2\pi)^4} = 2i \int [G^2(p)]_\omega \frac{d^4 p}{(2\pi)^4} + \frac{8\pi z^2 k_F^2}{(2\pi)^3 v_F}$$

the second term

$$\int \{G^2(p)\}_\omega P_{\alpha\beta;\alpha\beta}^{\omega}(p, q) \{G^2(q)\}_\omega \frac{d^4 q}{(2\pi)^4} \frac{d^4 p}{(2\pi)^4} - 2 \times \frac{2\pi i z^2}{v_F} \frac{k_F^2}{(2\pi)^4} \underbrace{\int \frac{d^4 q}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}^{\omega}(p, q) \{G^2(q)\}_\omega}_{\int d\Omega_p} \quad (*)$$

$$+ \left(\frac{2\pi i z^2}{v_F} \right)^2 \frac{(k_F^2)^2}{(2\pi)^4 (2\pi)^4} \int d\Omega_p d\Omega_q P_{\alpha\beta;\alpha\beta}^{\omega}(p, q) \quad (**)$$

the third one

$$- \frac{i k_F^2 z^2}{(2\pi)^3 v_F} \int \frac{d^4 p}{(2\pi)^4} \int d\Omega_{q'} \{ G^2(p) \}_w P_{\alpha\beta, \alpha\beta}^w(p, q') \left[\frac{v}{z} \frac{dk_F}{d\mu} - 1 \right] \quad (*)$$

$$- \frac{2\pi k_F^2 z^4}{(2\pi)^3 v_F^2} \frac{k_F^2}{(2\pi)^4} \int d\Omega_p \int d\Omega_{q'} P_{\alpha\beta, \alpha\beta}^w(p, q') \left[\frac{v}{z} \frac{dk_F}{d\mu} - 1 \right] \quad (**)$$

two (*) terms combine together

$2i \left(\frac{1}{z} - 1 \right)$

$$\frac{-i k_F^2 z^2}{(2\pi)^3 v_F} \int \frac{d^4 p}{(2\pi)^4} \int d\Omega_{q'} \{ G^2(p) \}_w P_{\alpha\beta, \alpha\beta}^w(p, q') \left[\frac{v}{z} \frac{dk_F}{d\mu} + 1 \right]$$

$$= \frac{2 k_F^2 z^2}{(2\pi)^3 v_F} \int d\Omega_{q'} \left(\frac{1}{z} - 1 \right) \left(\frac{v}{z} \frac{dk_F}{d\mu} + 1 \right) = \frac{8\pi k_F^2 z(1-z)}{(2\pi)^3 v_F} \left(\frac{v}{z} \frac{dk_F}{d\mu} + 1 \right)$$

two (**) terms combine together

$$\frac{-k_F^4 z^4}{(2\pi)^6 v_F^2} \int d\Omega_p d\Omega_{q'} P_{\alpha\beta, \alpha\beta}^w(p, q') \left[\frac{v}{z} \frac{dk_F}{d\mu} \right] = -4\pi \left[\frac{k_F^2 z^2}{(2\pi)^3 v_F} \right]^2$$

$$\otimes \int P_{\alpha\beta, \alpha\beta}^w(p, q') d\Omega_{q'} \left[\frac{v_F}{z} \frac{dk_F}{d\mu} \right]$$

$$\Rightarrow \frac{d(n)}{d\mu} = 2i \int \{ G^2(p) \}_w \frac{d^4 p}{(2\pi)^4} + \int [G^2(p)]_w P_{\alpha\beta, \alpha\beta}^w(p, q) [G^2(q)]_w \frac{d^4 p d^4 q}{(2\pi)^4 (2\pi)^4}$$

$$+ \frac{8\pi k_F^2 z^2}{(2\pi)^3 v_F} + \frac{8\pi k_F^2 (1-z)}{(2\pi)^3 v_F} v_F \frac{dk_F}{d\mu} - 4\pi \left(\frac{k_F^2 z^2}{(2\pi)^3 v_F} \right)^2 \int P_{\alpha\beta, \alpha\beta}^w(p, q) d\Omega_q$$

only this term survives

$$\left(\frac{v_F}{z} \frac{dk_F}{d\mu} \right)$$

The first two terms: $\Rightarrow 0$.

$$2i \int \frac{d^4 p}{(2\pi)^4} [G^2(p)]_\omega \int \frac{d^4 q}{(2\pi)^4} \left(1 - \frac{i}{2} \int \frac{d^4 \ell}{(2\pi)^4} P_{\alpha\beta;\alpha\beta}^w(p, q) [G^2(q)]_\omega \right)$$

$$= -2i \int \frac{d^3 p}{(2\pi)^3} \frac{\partial G}{\partial \omega} = -2i \int \left(G(\vec{p}, \omega = +\infty) - G(\vec{p}, \omega = -\infty) \right) \frac{d^3 p}{(2\pi)^3} = 0$$

this equals to the shift of total particle number due to shift of energy

zero. $(\epsilon \rightarrow \epsilon + u \Rightarrow \frac{\partial G}{\partial \epsilon} = -\frac{\partial G}{\partial u})$

The other 3-terms can be organized as

$$\frac{8\pi k_f^2 z}{(2\pi)^3 v_F} \left\{ 1 - \left(1 + \frac{k_f^2 z^2}{2(2\pi)^3 v_F} \int P_{\alpha\beta;\alpha\beta}^w(p, q) d\Omega_q \right) \left[v_F \frac{dk_f}{d\mu} \right] \right\}$$

$\leftarrow = 0$

$$+ \frac{8\pi k_f^2}{(2\pi)^3 v_F} v_F \frac{dk_f}{d\mu} = \frac{8\pi k_f^2}{(2\pi)^3} \frac{dk_f}{d\mu}$$

$$\Rightarrow \frac{dn}{d\mu} = \frac{8\pi k_f^2}{(2\pi)^3} \frac{dk_f}{d\mu} \Rightarrow n = \frac{8\pi k_f^3}{3(2\pi)^3} \Rightarrow k_f \text{ is the same as no-interacting system!}$$