1. Heisenberg spin chain

$$H = J \sum_{x=1}^{N} \left[S_x^1 S_{x+1}^1 + S_x^2 S_{x+1}^2 + \Delta \left(S_x^3 S_{x+1}^3 - \tfrac{1}{4} \right) \right]$$

$$= \frac{J}{2} \sum_{x=1}^{N} \left\{ S_x^+ S_{x+1}^- + S_x^- S_{x+1}^+ + 2\Delta \left[S_x^3 S_{x+1}^3 - \tfrac{1}{4} \right] \right\}$$

x: index of lattice sites

$$\vec{S}_x : \quad 1 \otimes 1 \cdots \otimes \underset{x}{\tfrac{1}{2}\vec{\sigma}} \otimes 1 \otimes \cdots 1$$

periodical boundary condition

$$\vec{S}_{x+N} = \vec{S}_x$$

total S^3 conserved : $[\sum_x S_x^3, H] = 0$.

For the states with fixed value $\sum_x S_x^3 = \frac{N}{2} - M$

$$|\psi\rangle = \sum_{x_1 < \cdots < x_M} \psi(x_1, \cdots x_M)\, S_{x_1}^- \cdots S_{x_M}^- \,|\underset{1}{\uparrow} \cdots \underset{N}{\uparrow}\rangle$$

$$\equiv \sum_{x_1 < \cdots < x_M} \psi(x_1 \cdots x_M)\, |x_1 \cdots x_M\rangle$$

assume that $|\psi\rangle$ is H's eigenstate : $H|\psi\rangle = E|\psi\rangle$.

① $\sum_{x} S_x^+ S_{x+1}^- \sum_{x_1<\cdots<x_M} \psi(x_1\cdots x_M)\,|\overset{\downarrow}{x_1}\cdots \overset{\downarrow}{x_M}\rangle$

shift $\underset{x}{\downarrow}\underset{x+1}{\uparrow} \to \underset{x}{\uparrow}\underset{x+1}{\downarrow}$,

$$= \sum_{x_1<\cdots x_M} \sum_{l=1}^{M} \psi(x_1\cdots x_l\cdots x_M)\ |x_1,\ldots x_l+1,\ldots x_M\rangle$$

$$= \sum_{x_1<\cdots x_M} \left\{ \sum_{l=1}^{M} \psi(x_1,\cdots x_l-1,\cdots x_M)\ |x_1\cdots x_l,\cdots x_M\rangle \right\}$$

index of spin "↓" position

Similarly

② $\sum_{x} \sum_{x_1<\cdots<x_M} S_x^- S_{x+1}^+ \psi(x_1\cdots x_M)\,|\overset{\downarrow}{x_1}\cdots \overset{\downarrow}{x_M}\rangle$

shift $\underset{x}{\uparrow}\underset{x+1}{\downarrow} \to \underset{x}{\downarrow}\underset{x+1}{\uparrow}$

$$= \sum_{x_1<\cdots<x_M} \sum_{l=1}^{M} \psi(x_1\, x_l\, x_M)\ |x_1\cdots x_l-1,\cdots x_M\rangle$$

$$= \sum_{x_1<\cdots<x_M} \left\{ \sum_{l=1}^{M} \psi(x_1\cdots x_l+1,\, x_M)\,|x_1\cdots x_l,\cdots x_M\rangle \right\}$$

③ $2\Delta \sum_{x} \sum_{x_1<\cdots x_M} (S_x^3\cdot S_{x+1}^3 - 1/4)\ \psi(x_1\cdots x_M)\ |x_1\cdots x_M\rangle$

$$= -\Delta \sum_{x_1<\cdots x_M} \underbrace{(\text{num of } \underline{\uparrow\downarrow} \text{ and } \underline{\downarrow\uparrow})}_{\text{define } n(x_1\cdots x_M)} \otimes \psi(x_1\cdots x_M)\ |x_1\cdots x_M\rangle$$

数目

From $H|\psi\rangle = E|\psi\rangle$

Compare the coefficient of the basis of $|x_1 \cdots x_M\rangle \Rightarrow$

$$\boxed{\frac{J}{2}\sum_{\ell=1}^{M}\left[\psi(x_1\cdots x_\ell+1,\ x_M) + \psi(x_1,\cdots,x_\ell-1,\cdots x_M)\right] - \frac{J}{2}\,\Delta\, n(x_1\cdots x_M)\ \psi(x_1\cdots x_M) = E\,\psi(x_1\cdots x_M)}$$

For every config of $\psi(x_1\cdots x_M)$

the number of terms in $\sum_{\ell=1}^{M}$ might be smaller than $2M$.

Say if x_ℓ and $x_{\ell+1}$ are neighbors, $x_{\ell+1} = x_\ell + 1$,

$$\psi(x_1\cdots \underset{\ell}{x_\ell+1},\ \underset{\ell+1}{x_{\ell+1}}\cdots) = \psi(\cdots \underset{\ell}{x_\ell+1},\ \underset{\ell+1}{x_\ell+1}\cdots)$$

should not exist, and will be omitted.

Bethe-ansartz

$$\boxed{\psi(x_1\cdots x_M) = \sum_P A_P\, e^{i\sum_{\ell=1}^{M} k_{P\ell}\, x_\ell}}$$

not necessary to satisfy boson permutation sym. due to ~~the confined~~

$$P = (P_1, P_2, \cdots P_M)$$

permutation of $1, 2, \cdots M$. ~~$x_1 < x_2$~~

$$1 \le x_1 < x_2 < \cdots x_M \le N.$$

Kind of we are studying boson problem, but we only consider the case of strict ordering.

where all the "$\downarrow$" are not neighbors. $n(x_1 \cdots x_M) = 2M$

$$\rightarrow \frac{J}{2}\sum_{l=1}^{M}\left[\psi(x_1 \cdots x_l+1, \cdots x_m) + \psi(x_1 \cdots x_l-1, \cdots x_M)\right]$$

$$- JM\Delta\,\psi(x_1 \cdots x_M) = E\,\psi(x_1 \cdots x_M)$$

Plug in the Bethe ansatz solution

$$\psi(x_1 \cdots x_l+1, \cdots x_M) + \psi(x_1 \cdots x_l-1, \cdots x_M)$$

$$= \sum_P A_P\, e^{i\sum_{l'=1}^{M} k_{Pl'} x_{l'}}\left[e^{ik_{Pl}} + e^{-ik_{Pl}}\right]$$

$$= \sum_P 2A_P \cos k_{Pl}\, e^{i\sum_{l'=1}^{M} k_{Pl'} x_{l'}}$$

$$\Rightarrow \sum_{l=1}^{M} \psi(x_1 \cdots, x_l+1, \cdots x_M) + \psi(x_1 \cdots x_l-1, \cdots x_M)$$

$$= \sum_{l=1}^{M}\sum_P 2A_P \cos k_{Pl}\, e^{i\sum_{l'=1}^{M} k_{Pl'} x_{l'}}$$

$$= \sum_P A_P \underbrace{\left(\sum_{l=1}^{M} 2\cos k_{Pl}\right)}_{\text{is independent of } P} e^{i\sum_{l'=1}^{M} k_{Pl'} x_{l'}}$$

$$\rightarrow = \left(2\sum_{l=1}^{M}\cos k_l\right)\psi(x_1 \cdots x_M) \Rightarrow \boxed{E = J\sum_{j=1}^{M}(\cos k_l - \Delta)}$$

"free part" add of the plane wave state.

Assume that a pair of $\overset{x_j}{\downarrow}\overset{x_{j+1}}{\downarrow}$ are neighbours, and others are not

$\sum_{\ell}{}' \cdots$ doesn't include

$$\psi(\cdots \overset{j}{\underset{\downarrow}{}} x_j+1, \overset{j+1}{\underset{\downarrow}{}} x_{j+1} \cdots) + \psi(\cdots \overset{j}{\underset{\downarrow}{}} x_j, \overset{j+1}{\underset{\downarrow}{}} x_{j+1}-1)$$

$$= \psi(\cdots x_{j+1}, x_{j+1} \cdots) + \psi(\cdots x_j, x_j)$$

this kind of term excluded!

now $n(x_1 \cdots x_M) = 2M-2$

$$\Rightarrow \frac{J}{2} \sum_{\ell}{}' \left[\psi(x_1 \cdots x_\ell+1, \cdots x_m) + \psi(x_1 \cdots x_{\ell-1}, x_m)\right]$$

$$☆_1 \quad -J(m-1)\,\Delta\, \psi(x_1 \cdots x_m) = E\,\psi(x_1 \cdots x_m)$$

we know Bethe ansatz solution $\psi(x_1 \cdots x_M) = \sum_P A_P\, e^{i\sum_{\ell=1}^{M} k_{P_\ell} x_\ell}$

literally satisfies

$$☆_2 \quad \frac{J}{2} \sum_{\ell} \psi(x_1 \cdots x_\ell+1, \cdots x_m) + \psi(x_1 \cdots x_\ell-1, x_m)$$

$$-J(m)\,\Delta\, \psi(x_1 \cdots x_M) = E\,\psi(x_1 \cdots x_m)$$

even keep $\boxed{x_{j+1} = x_j + 1}$

The difference between $☆_1$ and $☆_2 \Rightarrow$

$$\boxed{\frac{J}{2}\left(\psi(\cdots \underset{\overset{\uparrow}{j}}{x_{j+1}}, \underset{\overset{\uparrow}{j+1}}{x_{j+1}} \cdots) + \psi(\cdots \underset{\overset{\uparrow}{j}}{x_j}, \underset{\overset{\uparrow}{j+1}}{x_j} \cdots)\right) - J\,\Delta\,\psi(x_1 \cdots x_m) = 0}$$

describe interaction of two "$\downarrow\downarrow$"

Assume that three are three "$\downarrow$" are neighbours $\underset{j-1}{\downarrow}\ \underset{j}{\downarrow}\ \underset{j+1}{\downarrow}$

All other "$\downarrow$"'s are not adjacent to each other.

Then $\sum_{l} \cdots$ does not include the following "non-physical" terms

of

$$\psi(\cdots \underset{\substack{\uparrow\\ j-1}}{x_{j-1}+1}, \underset{\substack{\uparrow\\ j}}{x_j}, \underset{\substack{\uparrow\\ j+1}}{x_{j+1}} \cdots) + \psi(\cdots x_{j-1}, x_j-1, x_{j+1} \cdots)$$

$$+ \psi(\cdots x_{j-1}, x_j+1, x_{j+1}, \cdots) + \psi(\cdots x_{j-1}, x_j, x_{j+1}-1, \cdots)$$

$$= \psi(\cdots \underset{\substack{\uparrow\\ j-1}}{x_j}\ \underset{\substack{\uparrow\\ j}}{x_j} \cdots) + \psi(\cdots \underset{\substack{\uparrow\\ j-1}}{x_{j-1}}, \underset{\substack{\uparrow\\ j}}{x_{j-1}} \cdots) + \psi(\cdots \underset{\substack{\wedge\\ j}}{x_{j+1}}, \underset{\substack{\wedge\\ j+1}}{x_{j+1}} \cdots)$$

$$+ \psi(\cdots \underset{\substack{\uparrow\\ j-1}}{x_j}\ \underset{\substack{\uparrow\\ j}}{x_j} \cdots)$$

$\downarrow\downarrow\downarrow$ 4 domain walls missing

now $n(x_1 \cdots x_m) = 2M - 4$

The eigen equation $H\psi = E\psi$ goes to

$$\frac{J}{2} \sum_{l}{}' \left\{ \psi(\cdots \underset{\substack{\uparrow\\ l}}{x_l+1} \cdots) + \psi(\cdots \underset{\substack{\uparrow\\ l}}{x_l-1} \cdots) \right\} - J(m-2)\Delta\psi(x_1 \cdots x_m)$$

(unphysical terms excluded)

$$= E\,\psi(x_1 \cdots x_m)$$

(7)

the Bethe ansatz solution $\psi(x_1 \cdots x_M) = \sum_P A_P \, e^{i \sum_{l=1}^{M} K_{P_l} x_l}$ ↖ all the K's are different ↓ see below

satisfies $\frac{J}{2} \sum_l \{\psi(x_1 \cdots x_l + 1, \cdots x_m) + \psi(x_1 \cdots x_l - 1, \cdots x_m)\}$

$-J m \Delta \psi(x_1 \cdots x_m) = E \psi(x_1 \cdots x_m)$

⇒ The difference reads

$$\frac{J}{2}\left[2\psi(\cdots x_j, x_j \cdots) + \psi(\cdots x_{j-1} x_{j-1} \cdots) + \psi(\cdots x_{j+1} x_{j+1} \cdots)\right]$$

$$- 2J \Delta \psi(x_1 \cdots x_m) = 0$$

which is consistent with if we impose the condition

$$\frac{J}{2}[\psi(\cdots \underset{j-1}{x_{j-1}} \underset{j}{x_{j-1}} \cdots) + \psi(\cdots \underset{j-1}{x_j} \underset{j}{x_j} \cdots)] = J \Delta \psi(x_1 \cdots x_M)$$

& $$\frac{J}{2}[\psi(\cdots \underset{j}{x_j} \underset{j+1}{x_j} \cdots) + \psi(\cdots \underset{j}{x_{j+1}} \underset{j+1}{x_{j+1}} \cdots)] = J \Delta \psi(x_1 \cdots x_M)$$

This procedure can be generalized, Bethe ansatz solution satisfy the $H\psi = E\psi$, if the condition of

$$\boxed{\frac{1}{2}[\psi(\cdots \underset{j}{x_j} \underset{j+1}{x_j} \cdots) + \psi(\cdots \underset{j}{x_{j+1}} \underset{j+1}{x_{j+1}} \cdots)] = \Delta \psi(x_1 \cdots x_M)}$$

is satisfied for arbitrary config of $x_1 \cdots x_M$, and arbitrary j.

Ex: Assume the wavefunction $\psi(x_1 \cdots x_m)$ has two pairs $\downarrow\downarrow$, but they are not adjacent, prove that Bethe ansatz is still the solution

$$\uparrow\uparrow \cdots \downarrow\downarrow \cdots \uparrow\uparrow \cdots \downarrow\downarrow \cdots$$

Next we need from the "interaction relation" to determain the coefficient A_P.

Let us see an ~~example, of two different permutations~~

~~$P = (P_1, P_2, P_3 \quad)$ i.e. $P' = (1\ 2)\,P$~~

~~$P' = (P_2\ P_1\ P_3 \quad)$~~

set $j=1,\ j+1=2 \Rightarrow \frac{1}{2}\left[\psi(x_1 x_1 \cdots) + \psi(\cdot\, x_2 x_2 \cdots)\right] = \Delta\psi(x_1 \cdots x_m)$

$$\Rightarrow \frac{1}{2}\sum_P A_P\left[e^{ik_{p_1}x_1 + ik_{p_2}x_1} + e^{ik_{p_1}x_2 + ik_{p_2}x_2}\right] \otimes e^{i\sum_{j>2}k_{p_j}x_j} = \Delta\sum_P A_P\, e^{ik_{p_1}x_1 + k_{p_2}x_2} \otimes e^{i\sum_{j>2}k_{p_j}x_j}$$

$$\Rightarrow \sum_P A_P\left[e^{ik_{p_1}+ik_{p_2}} - 2\Delta e^{ik_{p_2}} + 1\right] \cdot e^{i\sum_{j>2}k_{p_j}x_j} = 0$$

we organize A_P into two classeds,

$P = (P_1\ P_2\ P_3 \cdots)$ and $P' = (P_2\ P_1 \cdots)$, i.e $P' = (1\ 2)\,P$

$$\Rightarrow \sum_{P,P'}{}^{\prime\, P}\left\{A_P\left[e^{ik_{p_1}+ik_{p_2}} - 2\Delta e^{ik_{p_2}} + 1\right] + A_{P'}\left[e^{ik_{p_2}+k_{p_1}} - 2\Delta e^{ik_{p_1}} + 1\right]\right\} \cdot e^{i\sum_{j>2}k_{p_j}x_j} = 0$$

$$\Rightarrow \quad \frac{A_{P'}}{A_P} = -\frac{e^{i(k_{P_1}+k_{P_2})} - 2\Delta e^{ik_{P_2}} + 1}{e^{i(k_{P_1}+k_{P_2})} - 2\Delta e^{ik_{P_1}} + 1}$$

where $P' = (1\ 2)P$, this is the two body scattering amplitude.

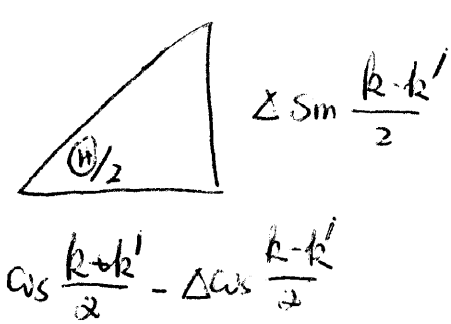
$$\boxed{\begin{array}{l} \text{if } k = k' \Rightarrow A_{P'} = -A_P \\ (A_P + A_{P'})\, e^{i(k x_j + k x_{j+1})} \\ = 0, \text{ these term cancels to zero} \end{array}}$$

generally if two permutations

$$\begin{array}{ccccc} k_{P_1} & k_{P_2} & \cdots & = & \cdots \overset{j}{k}\ \overset{j+1}{k'} \cdots \\ k_{P_1'} & k_{P_2'} & \cdots & = & \cdots k'\ k \cdots \end{array}$$

$$\Rightarrow \quad \boxed{\frac{A_{P'}}{A_P} = -\frac{e^{i(k+k')} - 2\Delta e^{ik'} + 1}{e^{i(k+k')} - 2\Delta e^{ik} + 1} \equiv -e^{i\Theta(k',k)}}$$

$$\Rightarrow \quad \frac{e^{i\frac{k+k'}{2}} - 2\Delta e^{-i\frac{k-k'}{2}} + e^{-i\frac{k+k'}{2}}}{e^{i\frac{k+k'}{2}} - 2\Delta e^{i\frac{k-k'}{2}} + e^{-i\frac{k+k'}{2}}} = \frac{\cos\frac{k+k'}{2} - \Delta\cos\frac{k-k'}{2} + i\Delta\sin\frac{k-k'}{2}}{\cos\frac{k+k'}{2} - \Delta\cos\frac{k-k'}{2} - i\Delta\sin\frac{k-k'}{2}}$$



$$\Rightarrow \quad \boxed{\Theta(k',k) = 2\tan^{-1}\frac{\Delta\sin\frac{k-k'}{2}}{\cos\frac{k+k'}{2} - \Delta\cos\frac{k-k'}{2}}}$$

Periodical boundary condition:

Set $x_1 \to x_1 + N$, and let all other x_j unmoved

$$\psi(x_1 \cdots x_m) = \psi(x_2, \cdots x_m, x_1+N)$$

$\uparrow$ x_1+N is the largest index.

$$\Rightarrow \sum_P A_P e^{ik_{P_1}x_1 + \cdots ik_{P_m}x_m} = \sum_P A_P e^{ik_{P_1}x_2 + k_{P_2}x_3 + \cdots ik_{P_M}x_1 + ik_{P_M}N}$$

define $P' = (P_2, P_3 \cdots P_m, P_1)$

$$\Rightarrow RHS = \sum_{P'} A_{P'} e^{ik_{P_2}x_2 + ik_{P_3}x_3 + \cdots ik_{P_1}x_1} e^{ik_{P_1} \cdot N}$$

$$\Rightarrow \boxed{A_{P'} e^{ik_{P_1}N} = A_P \quad \text{where } \left.\begin{array}{l} P = (P_1, P_2, \cdots P_N) \\ P' = (P_2, P_3 \cdots P_1) \end{array}\right\} \text{rotation}}$$

On the other hand $A_{P'} = -e^{i\Theta(k_{P_{j+1}}, k_{P_j})} A_P$ if $\begin{array}{l} P = (P_1 \cdots P_j P_{j+1} \cdots) \\ P' = (P_1 \cdots P_{j+1} P_j \cdots) \end{array}$

$$\Rightarrow A_{P'} = A_{P_2P_3 \cdots P_mP_1} = (-)e^{i\Theta(k_{P_m}, k_{P_1})} A_{P_2P_3 \cdots P_1P_m}$$

$$= (-)^2 e^{i\Theta(k_{P_m}, k_{P_1}) + i\Theta(k_{P_{m-1}}, k_{P_1})} A_{P_2P_3 \cdots P_1 P_{m-1} P_m}$$

$$= (-)^{m-1} e^{i\sum_{\ell=2,\cdots m} \Theta(k_{P_\ell}, k_{P_1})} A_{P_1 \cdots P_m}$$

$\Theta(k_{P_1}, k_{P_1}) = 0$

$$\Rightarrow \boxed{e^{ik_{P_1}N} = (-)^{m-1} e^{-i\sum_{\ell=1}^{m} \Theta(k_{P_\ell}, k_{P_1})}}$$

set $k_{P_1} = k_j$

generally $\Rightarrow$ $e^{ik_jN} = (-)^{M-1} e^{-i\sum_{l=1}^{M} \Theta(k_l, k_j)}$

$j = 1, \cdots M$

$$e^{i\Theta(k', k)} = \frac{1+e^{i(k+k')} - 2\Delta e^{ik'}}{1+e^{i(k+k')} - 2\Delta e^{ik}}$$

$$\boxed{\begin{aligned} k_j N &= 2\pi Q_j + \sum_{l=1}^{M} \Theta(k_j, k_l) \\ \text{where } Q_j &= \text{integer}, \text{ if } M \in \text{odd} \\ Q_j &= \text{half integer} \quad M \in \text{even} \end{aligned}}$$

Bethe ansatz equation!

For a given quantum number $(Q_1, Q_2 \cdots Q_M)$

solve the value of $(k_1, \cdots k_M) \longrightarrow E = J\sum_{j=1}^{M} (\cos k_j - \Delta)$

at $\Delta = 1$, we parameterize

$$e^{ik_j} = \frac{\lambda_j + i/2}{\lambda_j - i/2} \Rightarrow \boxed{\lambda_j = \frac{1}{2}\cot\frac{k_j}{2}}$$

$$e^{i\Theta(k_j, k_l)} = \frac{1+e^{i(k_j+k_l)} - 2e^{ik_j}}{1+e^{i(k_j+k_l)} - 2e^{ik_l}} = \frac{1 + \dfrac{(\lambda_j + i/2)(\lambda_l + i/2) - 2(\lambda_j + i/2)(\lambda_l - i/2)}{(\lambda_j - i/2)(\lambda_l - i/2)}}{\dfrac{(\lambda_j - i/2)(\lambda_l + i/2) + (\lambda_j + i/2)(\lambda_l + i/2) - 2(\lambda_l + i/2)(\lambda_j - i/2)}{(\lambda_j - i/2)(\lambda_l - i/2)}}$$

$$= \frac{2\lambda_j\lambda_l - \frac{1}{2} - 2(\lambda_j\lambda_l + \frac{i}{2}\lambda_l - \frac{i}{2}\lambda_j - 1/4)}{2\lambda_j\lambda_l - 1/2 - 2(\lambda_j\lambda_l - \frac{i}{2}\lambda_l + \frac{i}{2}\lambda_j - 1/4)} = \frac{-i(\lambda_l - \lambda_j) - 1}{i(\lambda_l - \lambda_j) - 1} = (-)\left(\frac{\lambda_l - \lambda_j - i}{\lambda_l - \lambda_j + i}\right)$$

the Bethe-Ansatz equation is

$$\boxed{\left(\frac{\lambda_j+\frac{i}{2}}{\lambda_j-\frac{i}{2}}\right)^N = \prod_{\substack{l=1\\(l\neq j)}}^{M} \frac{\lambda_j-\lambda_l+i}{\lambda_j-\lambda_l-i}}$$

example: $M=2 \Rightarrow$

$$\left(\frac{\lambda_1+i/2}{\lambda_1-i/2}\right)^N = \frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}, \quad \left(\frac{\lambda_2+i/2}{\lambda_2-i/2}\right)^N = \frac{\lambda_2-\lambda_1+i}{\lambda_2-\lambda_1-i}$$

Set $\cos\theta_j = \dfrac{\lambda_j}{\sqrt{\lambda_j^2+1/4}}$ $\quad \sin\theta_j = \dfrac{1/2}{\sqrt{\lambda_j^2+1/4}}$

$$\Rightarrow \boxed{e^{2N\theta_1 i} = \frac{\lambda_1-\lambda_2+i}{\lambda_1-\lambda_2-i}}$$

bound state solution

if θ_1 is complex number, as $N\to+\infty$, LHS $\to 0$, or ∞

$\Rightarrow \lambda_1-\lambda_2 = \mp i$, and if λ_1, λ_2 are solutions

$\Rightarrow \lambda_1^*, \lambda_2^*$ are also solutions

we can guess these two the same k_1+k_2

$\lambda_1 = x+\frac{i}{2}$

$\lambda_2 = x-\frac{i}{2}$

$$\Rightarrow e^{i(k_1+k_2)} = \frac{x+i}{x}\cdot\frac{x}{x-i} = \frac{x+i}{x-i}$$

$$\Rightarrow \cos(k_1+k_2) = \frac{x^2-1}{x^2+1}$$

$$\Rightarrow E = J(\cos k_1 + \cos k_2 - 2) \qquad e^{ik_1} = \frac{\lambda_1 + \frac{i}{2}}{\lambda_1 - i/2} = \frac{x+i}{x}$$

$$= J\left[1 + \frac{x^2}{x^2+1} - 2\right] = -\frac{J}{x^2+1} \qquad e^{ik_2} = \frac{\lambda_2 + i/2}{\lambda_2 - i/2} = \frac{x}{x-i}$$

$$= \frac{J}{2}\left[\cos(\underbrace{k_1+k_2}_{k}) - 1\right] \leftarrow \text{bound state energy.}$$

Consider scattering states in which k_1, k_2 are real.

$$\frac{\cos k - 1}{2} \geq (\cos k_1 - 1) + \cos(k_2 - 1) \quad \text{it canbe proved.}$$

~~set y_i = $\frac{1 - \cos k_i}{2}$ = $\sin\frac{k_i}{2}$~~

$\Rightarrow$ For $J < 0$, FM ~~states~~ chain), the solution of the complex momentum has lower energy

$J > 0$ (AFM) the magnon bounds have higher energy.

For AFM chain, ground state, λ_i are real.

Quantum numbers in the ground state: (if we set $\Delta=0$)

$$e^{ik_j N} = (-)^{M-1} \Rightarrow k_j = \frac{2\pi Q_j}{N}, \quad \begin{array}{l} Q_j = \text{integer for } M \text{ odd} \\ Q_j = \text{half integer for } M \text{ even} \end{array}$$

we set N : even

free system

$$E = J\sum_{j=1}^{M}(\cos k_j - 1) = -\frac{J}{2}\sum_{j=1}^{M}\frac{1}{\lambda_j^2+1/4} \qquad e^{ik_j} = \frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}}$$

$$\cdot\ \lambda_j = \frac{1}{2}\operatorname{ctg}\frac{k_j}{2}, \quad k_j = \frac{2\pi Q_j}{N} \in \left[-\frac{\pi M}{N}, +\frac{\pi M}{N}\right]. \qquad \cos k_j = \operatorname{Re} e^{i\lambda_j} = \frac{\lambda_j^2 - 1/4}{\lambda_j^2+1/4}$$

For the ground state $Q_j = \underbrace{-\frac{M-1}{2}, -\frac{M-3}{2}, \cdots \frac{M-1}{2}}_{M}$

if M is odd, $\Rightarrow$ "0" is included

even $\Rightarrow$ "0" is not included

$$\Rightarrow \cancel{E = -\frac{J}{2}\sum_{j=1}^{M}\frac{4}{\operatorname{ctg}^2\frac{k_j}{2}+1} = -2J\sum_{j=1}^{M}} \quad \cancel{k_j \in -\frac{\pi}{N}(M-1), \ldots \frac{\pi}{N}(M-1)}$$

Now put interaction Δ

$$N k_j = 2\pi Q_j \longrightarrow 2\pi Q_j + \sum_{l=1}^{N/2} \Theta(k_j - k_l) \qquad \leftarrow M = N/2$$

the distance between adjacent k_j changes $\leftarrow$ density of states in momentum space is not uniform!

2. δ-interaction of boson

$$H = \int dx \left(\frac{\partial u^\dagger}{\partial x} \frac{\partial u}{\partial x} + c\, u^\dagger u^\dagger u\, u \right)$$ second Quantization

$$[\, u(x,t),\ u^\dagger(x',t) \,] = \delta(x-x')$$

set $|\psi\rangle = \int dx_1 \cdots dx_N\ \psi(x_1 \cdots x_N, t)\ u^\dagger(x_1) \cdots u^\dagger(x_N) |0\rangle$

$\rightarrow$ δ interaction

$$\boxed{ -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \psi + 2c \sum_{i<j} \delta(x_i - x_j)\, \psi = E\, \psi }$$ first Quantizatio

Assume $c>0$, and looking for Bethe ansatz Solution.

example of $N=2$

$$-\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \psi + 2c\, \delta(x_1 - x_2)\, \psi = E\, \psi$$

ψ is continous but derivtive is discontinous in

$$\psi = \theta(x_2 - x_1)\, \psi_{12}(x_1, x_2) + \theta(x_1 - x_2)\, \psi_{21}(x_1, x_2)$$

Bose statistics $\psi(x_1\, x_2) = \psi(x_2\, x_1)$

$$\Rightarrow \psi_{12}(x_1, x_2) = \psi_{21}(x_2, x_1)$$

For $x_1 < x_2$, $\psi = A_{12}\, e^{ik_1x_1 + ik_2x_2} + A_{21}\, e^{ik_2x_1 + ik_1x_2}$

($k_1 \neq k_2$, you can prove if $k_1 = k_2$, then $\psi \equiv 0$)

if $x_2 < x_1$, $\psi = A_{12}\, e^{ik_2x_1 + ik_1x_2} + A_{21}\, e^{ik_1x_1 + ik_2x_2}$

set relative coordinate / center of mass coordinate

$$y = x_2 - x_1, \quad X = \frac{x_1 + x_2}{2}, \quad K = k_1 + k_2$$

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} = \frac{1}{2}\frac{\partial^2}{\partial X^2} + 2\frac{\partial^2}{\partial y^2}$$

$$\psi = \begin{cases} e^{iKX}\left(A_{12}\, e^{i(k_2-k_1)y/2} + A_{21}\, e^{-i(k_2-k_1)y/2}\right) & y>0 \\ e^{ikX}\left(A_{12}\, e^{-i(k_2-k_1)y/2} + A_{21}\, e^{i(k_2-k_1)y/2}\right) & y<0 \end{cases}$$

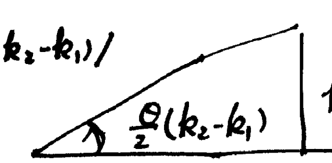
in the center of mass coordinate

$$-2\frac{\partial^2}{\partial y^2}\psi + 2c\,\delta(y)\,\psi = E\,\psi$$

$$\rightarrow \quad \left.\frac{\partial\psi}{\partial y}\right|_{0^+} - \left.\frac{\partial\psi}{\partial y}\right|_{0^-} = c\,\psi\Big|_0$$

$$\Rightarrow \quad i(k_2-k_1)(A_{12} - A_{21}) = c\,(A_{12} + A_{21})$$

$$\Rightarrow \quad \frac{A_{21}}{A_{12}} = -\frac{c + i(k_1-k_2)}{c - i(k_1-k_2)} = -e^{i\theta(k_2-k_1)/}$$



$$\frac{1}{2}\theta(k_2-k_1) = \tan^{-1}\frac{k_1-k_2}{2} \quad \text{or} \quad \boxed{\theta(k_2-k_1) = -2\tan^{-1}\frac{k_2-k_1}{c}}$$

Now we consider $N \geq 3$, define permutation $Q = (Q_1, Q_2, \cdots Q_N)$

$$\boxed{\psi = \sum_Q \theta(x_{Q_1} < x_{Q_2} < \cdots x_{Q_N})\, \psi_Q(x_1, \cdots x_N)}$$

Boson statistics $\longrightarrow$ $\underset{Q=E}{\psi}(x_{Q_1} \cdots x_{Q_N}) = \psi_Q(x_1 \cdots x_N)$

if we know $\psi_{12\cdots N}(x_1 \cdots x_N)$, then we know everything.

For example $\psi_{23\cdots N1}(x_1, \cdots x_N) = \psi_{12\cdots N}(x_2\ x_3 \cdots x_N, x_1)$

Bethe ansatz: for $x_1 < x_2 < \cdots < x_N$

$$\psi = \psi_{12\cdots N} = \sum_P A_P\, e^{ik_{P_1} x_1 + ik_{P_2} x_2 + \cdots ik_{P_N} x_N},$$

where $P = (P_1, P_2, \cdots P_N)$, $k_1 \cdots k_N$ are $\underbrace{\text{real numbers}}_{\text{unequal}}$.

In the region of $x_2 < \cdots < x_N < x_1$

$$\psi = \psi_{23\cdots N1} = \sum_P A_P\, e^{ik_{P_1} x_2 + ik_{P_2} x_3 + \cdots + ik_{P_N} x_1}.$$

Similarly to case of $N = 2$, for the two permutation

$$k_{P_1}\ k_{P_2} \cdots k_{P_N} = \cdots \overset{j}{k}\ \overset{j+1}{k'} \cdots$$

$$k_{P'_1}\ k_{P'_2} \cdots k_{P'_N} = \cdots k'\ k \cdots$$

i.e $(P_1\ P_2 \cdots P_N) = (j, j+1)(P_1'\ P_2' \cdots P_N')$

Consider
$$\psi_{123\cdots N} = \sum_P{}' \Big\{ A_P\, e^{i k_{P_1} x_1 + i k_{P_2} x_2 + \cdots \underline{i k_{P_j} x_j + i k_{P_{j+1}} x_{j+1}} + \cdots} + A_{P'}\, e^{i k_{P_1} x_1 + i k_{P_2} x_2 + \cdots \underline{i k_{P_{j+1}} x_j + i k_{P_j} x_{j+1}} + \cdots} \Big\}$$

$$\psi_{123\cdots j+1, j \cdots N} = \sum_P{}' \Big[A_P\, e^{i k_{P_1} x_1 + \cdots + \underline{i k_{P_j} x_{j+1} + i k_{P_{j+1}} x_j} + \cdots} + A_P'\, e^{i k_{P_1} x_1 + \cdots \underline{i k_{P_{j+1}} x_{j+1} + i k_{P_j} x_j} + \cdots} \Big]$$

introducing the collective coordinate between x_j and x_{j+1} as X and y

$$\psi = \begin{cases} \sum_P e^{i k_{P_1} x_1 + \cdots} \left[A_P\, e^{i(k_{P_{j+1}} - k_{P_j}) y/2} + A_{P'}\, e^{-i(k_{P_{j+1}} - k_{P_j}) y/2} \right] e^{iKX} & (y>0) \\ \sum_P e^{i k_{P_1} x_1 + \cdots} \left[A_P\, e^{-i(k_{P_{j+1}} - k_{P_j}) y/2} + A_{P'}\, e^{i(k_{P_{j+1}} - k_{P_j}) y/2} \right] e^{iKX} & (y<0) \end{cases}$$

The continuity relation
$$\left.\frac{\partial \psi(x_1 \cdots y)}{\partial y}\right|_{0^+} - \left.\frac{\partial \psi(x_1 \cdots y, X \cdots)}{\partial y}\right|_{0^-} = c\, \psi(x_1 \cdots y, X)\Big|_{y=0}$$

$\Rightarrow$ For each P and its partner $P' = (j, j+1) P$, we ~~m~~ need

$$i(k_{P_{j+1}} - k_{P_j})(A_P - A_{P'}) = c(A_P + A_{P'})$$

$$\Rightarrow \boxed{\frac{A_{P'}}{A_P} = -\frac{c + i(k_{P_{j+1}} - k_{P_j})}{c - i(k_{P_{j+1}} - k_{P_j})} = -e^{i\theta(k_{P_{j+1}} - k_{P_j})} \quad \text{and} \quad \theta(k_{P_{j+1}} - k_{P_j}) = -2\tan^{-1}\frac{k_{P_{j+1}} - k_{P_j}}{c}}$$

Let us consider periodic boundry condition:

$$\psi(x_1=0, x_2 \cdots x_N) = \psi(L, x_2, \cdots x_N)$$

where $x_1 < x_2 < \cdots x_N$ $\qquad\qquad$ $x_2 < x_3 < \cdots < x_N < x_1$

$$\psi_{2\cdots N1}(x_1 x_2 \cdots x_N) = \sum_P A_{P_1\cdots P_N}\, e^{ik_{P_1}x_2 + k_{P_2}x_3 + \cdots ik_{P_N}x_1}$$

$$= \sum_P A_{P_2\cdots P_N P_1}\, e^{ik_{P_2}x_2 + \cdots ik_{P_1}x_1}$$

$$\Rightarrow \psi_{12\cdots N}(0, x_2 \cdots x_N) = \psi_{2\cdots N1}(L, x_2 \cdots x_N)$$

$$\sum_P A_{P_1\cdots P_N}\, e^{ik_{P_1} + ik_{P_2}x_2 + \cdots} = \sum_P A_{P_2\cdots P_N P_1}\, e^{ik_{P_1}L + ik_{P_2}x_2 + \cdots}$$

$$\Rightarrow \boxed{A_{P_1\cdots P_N} = A_{P_2\cdots P_N P_1}\, e^{ik_{P_1}L}}$$

$\cancel{A_{P_1}\cdots A_{P_N}}$

$$\frac{A_{P_2P_1\cdots P_N}}{A_{P_1P_2\cdots P_N}} \cdot \frac{A_{P_2P_3P_1\cdots P_N}}{A_{P_2P_1P_3\cdots P_N}} \cdot \frac{A_{P_2P_3P_4P_1\cdots P_N}}{A_{P_2P_3P_1\cdots P_N}} \cdots \frac{A_{P_2\cdots P_NP_1}}{A_{P_2P_3\cdots P_1P_N}} = \frac{A_{P_2\cdots P_NP_1}}{A_{P_1P_2\cdots P_N}} = e^{-ik_{P_1}L}$$

$$(-)^{N-1}\, e^{i\theta(k_{P_2}-k_{P_1})}\, e^{i\theta(k_{P_3}-k_{P_1})} \cdots e^{i\theta(k_{P_N}-k_{P_1})} = e^{-ik_{P_1}L}$$

or $\quad e^{ik_{P_1}L} = (-)^{N-1}\, e^{+i\sum_{j=1}^{N}\theta(k_{P_1}-k_{P_j})}$

Set $k_{P_1} = i$

$$\Rightarrow \boxed{e^{ik_iL} = (-)^{N-1}\, e^{+i\sum_{j=1}^{N}\theta(k_i-k_j)}}$$ ← Bethe ansatz Eq.

$$\boxed{k_iL = 2\pi I_i + \sum_{j=1}^{N}\theta(k_i-k_j), \quad I_i = \begin{cases} \text{integer}, & N=\text{odd} \\ \text{half integer} & N=\text{even} \end{cases}}$$

If we can solve k_i, $i=1,2,\cdots N$, then we can find $E=\sum_{j=1}^{N} k_j^2$.