Lect 9

Landau Fermi liquid theory (I)

§ Concept of quasi-particles

Suppose we have a many-body fermion ground state $|G\rangle$. At time $t=0$, an extra particle is added at the plane wave state $C_k^\dagger$. After a time interval of T, we check what's the amplitude remaining in this state.

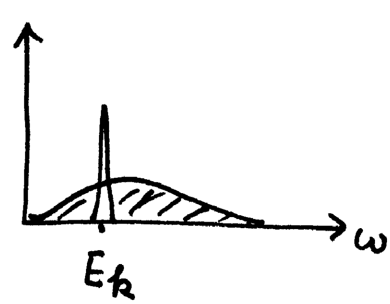
$$G_k(t) = \langle G| e^{iHT} C_k \, e^{-iHT} C_k^\dagger |G\rangle = \langle G| C_k(T) \, C_k^\dagger(0) |G\rangle$$

in terms of Lehmann's representation

$$G_k(t) = \sum_m \langle G| C_k(T)|m\rangle\langle m| C_k^\dagger(0) |G\rangle = \sum_m |\langle G|C_k|m\rangle|^2 \, e^{-i(E_m - E_g)T}$$

In the Fermi liquid state, the distribution of the spetral weight has a special value of δ-peak (may be brodened), and a continuum (incoherent part)

$$Z = |\langle \Psi_k | C_k | G\rangle|^2 \leftarrow$$ wavefunction renormalization factor



$$\Rightarrow G_k(t) = Z \, e^{-iE_k' T} \leftarrow$$ quasi-particle

$$+ \int \frac{d\omega}{2\pi} A(\omega) \, e^{-i\omega t}$$

"$0 < Z < 1$" justifies the validity of Fermi liquid state. Even though in an interacting system, we can still look it as if a free system. Quasi-particle is like "a running horse running in a dusty road, dressed by a cloud of dust

§ Physical content of the self-energy P_{249} Negele & Orland.

no-interacting Green function $\quad G_0(k,\omega) = \dfrac{1}{\omega - \varepsilon_k + i\,\mathrm{sgn}(\omega)\eta} \rightarrow$ single pole with residue "1".

For the full green's function

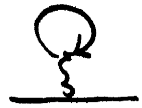
$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma(k,\omega)},$$

we need to exam the structures of poles and the residues.

Let us only exam the first two order pertubation theory

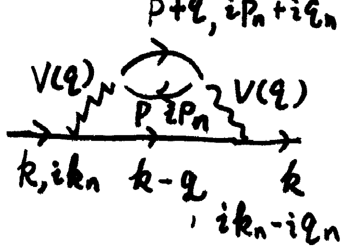
$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma_1(k) - \Sigma_2(k,\omega)}$$

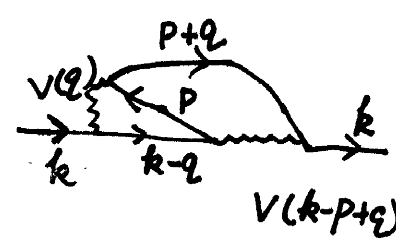
The first order is HF which is frequency independent

$\Sigma_1(k) =$  $+$  $= \sum_{k'} v(q=0)\, n_{k'} - V(k-k')\, n_{k'}$.

(v has no fregency dependence $\rightarrow$ $\Sigma_1(k)$ has no frequency dependence.

The second order

$\Sigma_2(k,\omega) =$ 

$+$ 

frequency summation

$$\frac{1}{\beta}\sum_{iq_n}\left[\frac{1}{\beta}\sum_{ip_n}\frac{1}{ip_n+iq_n-\varepsilon_{p+q}}\,\frac{1}{ip_n-\varepsilon_p}\right]\frac{1}{ik_n-iq_n-\varepsilon_{k-q}}$$

$$= \frac{1}{\beta}\sum_{iq_n}\frac{n_f(\varepsilon_p)-n_f(\varepsilon_{p+q})}{iq_n-(\varepsilon_{p+q}-\varepsilon_p)}\;\frac{1}{ik_n-iq_n-\varepsilon_{k-q}}$$

define $S = \frac{1}{\beta} \sum_{iq_n} \frac{1}{iq_n - (\varepsilon_{p+q} - \varepsilon_p)} \frac{1}{ik_n - iq_n - \varepsilon_{k-q}}$

& $f(z) = \frac{1}{z - (\varepsilon_{p+q} - \varepsilon_p)} \frac{1}{ik_n - z - \varepsilon_{k-q}}$

$I = \lim_{R\to\infty} \int \frac{dz}{2\pi i} f(z) \frac{1}{e^{\beta z} + 1} = 0. \Rightarrow$

$$\frac{1}{\beta} \sum_n f(iq_n) + \frac{1}{e^{\beta(\varepsilon_{p+q} - \varepsilon_p)} - 1} \frac{1}{ik_n - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})}$$

$$+ \frac{1}{e^{\beta(ik_n - \varepsilon_{k-q})} - 1} \frac{-1}{ik_n - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})} = 0$$

$$\Rightarrow \frac{1}{\beta} \sum_n f(iq_n) = - \frac{[n_B(\varepsilon_{p+q} - \varepsilon_p) + 1 - n_f(\varepsilon_{k-q})]}{ik_n - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})}$$

$$\Rightarrow \Sigma_2(k,\omega) = \sum_p \sum_q \frac{[n_f(\varepsilon_p) - n_f(\varepsilon_{p+q})][1 - n_f(\varepsilon_{k-q}) + n_B(\varepsilon_{p+q} - \varepsilon_p)]}{\omega + i\eta - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})} [V^2(q) - V(q)V(k-p+q)]$$

real frequency

$\omega > 0$ for $k > k_f$.

$\Sigma_2(k,\omega)$ explicitly depends on ω. Σ_2 has an infinite number of poles at $\omega = \varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q} \longrightarrow$ finite imaginary part.

☆ Let us consider a simple example. If $\Sigma_2(k,\omega)$ has two poles

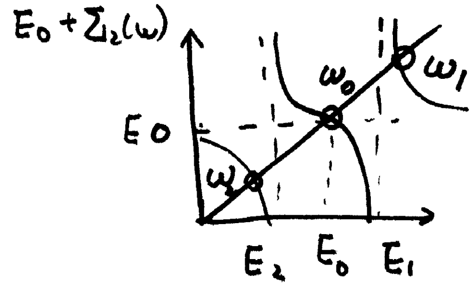
$$\Sigma_2(\omega) = \frac{A_1}{\omega - E_1 + i\eta} + \frac{A_2}{\omega - E_2 + i\eta},$$ then what's the

poles and residues of $G(\omega) = \frac{1}{\omega - E_0 - \Sigma_2(\omega)}$

we assume E_1 is above E_0, and E_2 is below E_0, and the residues in Σ_2 are very small, satisfying $\frac{A_1}{|E_1-E_0||E_2-E_0|}, \frac{A_2}{|E_1-E_0||E_2-E_0|} \ll 1$.

Since Σ_2 is small at $\omega \neq E_1, E_2$, we first consider Σ_1, which gives the order of the pole of $\omega = E_0$. Including Σ_2, we solve $\omega = E_0 + \Sigma_2(\omega)$

ω_0 is close to E_0; $\omega_{1,2}$ close to $E_{1,2}$.



We expand $G(\omega)$ around each poles

$$\omega_i = E_0 + \Sigma_2(\omega_i) \qquad i = 0, 1, 2$$

$$\omega - E_0 - \Sigma_2(\omega) = \omega - E_0 - \Sigma_2(\omega_i) + (\omega - \omega_i)\Sigma_2'(\omega_i)$$

$$= (\omega - \omega_i)(1 - \Sigma_2'(\omega_i)) \Rightarrow G(\omega) \simeq \frac{1}{1-\Sigma_2'(\omega)} \frac{1}{\omega - \omega_i + i\eta}$$

$$\frac{1}{1-\Sigma_2'(\omega_0)} = \frac{1}{1 + \sum_i \frac{A_i}{(\omega_0 - E_i)^2}} \approx 1 - \sum_{i=1}^{2} \frac{A_i}{(E_0 - E_i)^2}$$

$$\frac{1}{1-\Sigma_2'(\omega_1)} = \frac{1}{1 + \frac{A_1}{(\omega_1 - E_1)^2} + \frac{A_2}{(\omega_1 - E_2)^2}} \approx \frac{(\omega_1 - E_1)^2}{A_1}$$

consider $\omega_1 = E_0 + \frac{A_1}{\omega_1 - E_1} + \frac{A_2}{\omega_1 - E_2}$

$$\Rightarrow \omega_1 - E_0 \approx \frac{A_1}{\omega_1 - E_1} \quad \text{or} \quad \omega_1 - E_1 \approx \frac{A_1}{\omega_1 - E_0}$$

$$\Rightarrow \frac{1}{1-\Sigma_2'(\omega_1)} \approx \frac{A_1}{(E_1 - E_0)^2}, \quad \text{similarly} \quad \frac{1}{1-\Sigma_2'(\omega_2)} \approx \frac{A_2}{(E_2 - E_0)^2}.$$

After switching on interaction, the single pole is fragmented into three poles. The major one is still close to E_0, but with a smaller residue $1 - \sum_i \frac{A_i}{(E_0 - E_i)^2}$. This pole is called the quasi-particle pole.

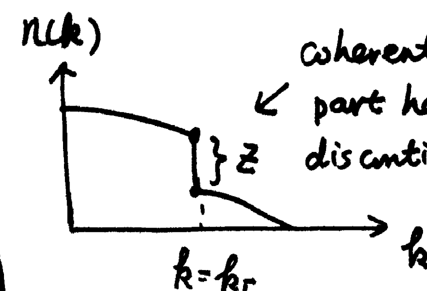
The ~~strength~~ removed from the quasi-particle pole has been redistributed to poles at $\omega_{1,2}$. These poles represent complicated many-body medium excitations of

Now consider the real case.

$$\boxed{\left.\frac{1}{1 - \frac{\partial}{\partial\omega}\Sigma_2(\omega)}\right|_{\omega \simeq E_0} \approx 1 - \sum_{p,q} \frac{A_{p,q}^2}{[E_0 - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{p-q})]^2} = Z}$$

depleted to the incoherent background.

The Landau fermi-liquid is based on the assuption that Z remains finite. →



$$c_k^\dagger c_k = \underbrace{Z}_{\text{WF renormalization}} \underbrace{\tilde{c}_k^\dagger \tilde{c}_k^\dagger}_{\text{quasi-particle}} + \cdots$$

Now let's consider damping

$$\frac{1}{\tau} = -2\,\mathrm{Im}\,\Sigma_2 = 2\pi \sum_{p,q} \underbrace{|A_{p,q}|^2}\, \delta[\omega - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})]$$

$$\frac{1}{\tau} \approx |V|^2 \int_0^{\varepsilon_k} \underbrace{d\epsilon_{k-q}}_{\text{particle}} \int_0^{\epsilon_k - \epsilon_{k-q}} \underbrace{d\epsilon_{p+q}}_{\text{particle}} \; \rho(\epsilon_{k-q})\, \rho(\epsilon_{p+q})\, \rho(\underbrace{\varepsilon_p}_{\text{hole}} = -\varepsilon_k + (\varepsilon_{p+q} + \varepsilon_{k-q}))$$

$$\leq |V|^2 \rho_{max}^3 \varepsilon_k^2 \quad \Longrightarrow \quad \boxed{\tau(\epsilon) \propto |\epsilon - \epsilon_F|^{-2}}$$

☆ effective mass.

$$\mathcal{E} = \frac{k^2}{2m} - \mu + \Sigma(k, \mathcal{E})$$

$$\frac{d\epsilon}{dk} = \frac{k}{m} + \frac{\partial\Sigma}{\partial k} + \frac{\partial\Sigma}{\partial\mathcal{E}}\frac{d\mathcal{E}}{dk} \Rightarrow \frac{d\mathcal{E}}{dk} = \left(1 - \frac{\partial\Sigma}{\partial\epsilon}\right)^{-1}\left(\frac{k}{m} + \frac{\partial\Sigma}{\partial k}\right)$$

$$= \left(1 - \frac{\partial\Sigma}{\partial\epsilon}\right)^{-1}\frac{k}{m}\left(1 + \frac{m}{k}\frac{\partial\Sigma}{\partial k}\right)$$

define $\frac{d\epsilon}{dk} = \frac{k}{m^*}$ $\Rightarrow$ $\boxed{m^* = m\left(1 + \frac{m}{k}\frac{\partial\Sigma}{\partial k}\right)^{-1} \times \left(1 - \frac{\partial\Sigma}{\partial\omega}\right)\Big|_{\omega=\mathcal{E}}}$