Lect 7 Boltzmann transport (I)

§ Wigner distribution

we need to consider a spatially inhomogenous (slowly varing) system. We will talk about the particle with momentum $\vec{p}$ at point $\vec{r}$, and the distribution function $n_{\alpha\beta}(p;r,t)$ in the phase space. A more rigours definition is as follows

define $f(p;k;t) = \langle C^\dagger_{p+k/2} C_{p-k/2} \rangle (t)$, and perform the Fourier transform over small variable k, and arrive at

$$n_{\alpha\beta}(p;r,t) = \sum_k f(p;k;t)\, e^{ikr}$$

another way to express $n_{\alpha\beta}(p;r,t) = \int dr' e^{ip\cdot r'} \langle \psi^\dagger_\alpha(r+\frac{r'}{2})\, \psi_\beta(r-\frac{r'}{2}) \rangle$

where "r" is the center of mass coordinate, r' is the ralative coordinate.

$n_{\alpha\beta}(p;r,t)$ is a semiclassical distribution. The resolution of Δp and Δr need to satisfy $\Delta p \cdot \Delta r \geq \hbar/2$.

§ Boltzmann equation:

Let us study the equation of motion of $n(p;r;t)$. It has three contributions: ① Flow in the momentum space

② Flow in the real space

③ Collisions

$$\frac{\partial n(p;r;t)}{\partial t} + \nabla_r [\underset{\uparrow\, dr/dt}{v_p(r,t)}\, n(p;r,t)] + \nabla_p [\underset{\uparrow\, dp/dt}{f_p(r,t)}\, n(p;r,t)] = I_{collision}$$

where $v_p(r,t) = \nabla_p \varepsilon_p(r,t)$, $f_p(r,t) = -\nabla_r \varepsilon_p(r,t)$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} n(p;r;t) + \nabla_p \varepsilon(p,r,t)\, \nabla_r n(prt) - \nabla_r \varepsilon(prt)\, \nabla_p n(prt) = I_{collision}}$$

linearing the equation:

$$\varepsilon(p,r,t) = \varepsilon_0(p,r,t) + \frac{1}{V}\sum_{p'} f(p,p')\, \delta n(p'rt)$$

$$n(p;r,t) = n_0(p,r,t) + \delta n(prt)$$

$n_0(p,r,t)$ is a slow varying function of r. but has a sharp discontinuity in momentum space as a function of P.

$$\Rightarrow \frac{\partial}{\partial t}\delta n(prt) + \vec{v}_p \cdot \nabla_r \delta n(prt) - \nabla_p n^0(prt) \left(\sum_{p'} f_{pp'}\, \nabla_r \delta n(p'rt)\right) = I_{coll}$$

$$\frac{\partial}{\partial t}\delta n(prt) + \vec{v}_p \cdot \nabla_r \left[\delta n(prt) - \frac{\partial n^0(p r,t)}{\partial \varepsilon_p} \sum_{p'} f_{pp'}\, \delta n(p'rt)\right] = I_{coll}$$

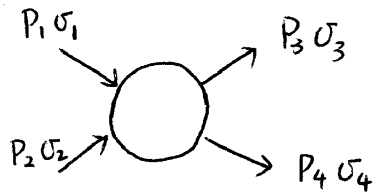
do Fourier transform over vairable $rt \to q, \omega$

$$\delta n(p\, r\, t) = \sum_q n(p;q,\omega)\, e^{i(qr-\omega t)}$$

$$\Rightarrow (-i\omega + i\vec{v}_p\cdot\vec{q})\, \delta n(pq\omega) - \frac{\partial n^0}{\partial \varepsilon_p} \vec{v}_p \cdot i\vec{q} \cdot \left(\sum_{p'} f_{pp'}\, \delta n(p'q\omega)\right) = I_{coll}$$

$$(\omega - vq\cos\theta)\, \delta n(pq\omega) + \frac{\partial n^0}{\partial \varepsilon_p} vq\cos\theta \left(\sum_{p'} f_{pp'}\, \delta n(p'q\omega)\right) = i I_{coll}$$

Collision integral



$$\frac{2\pi}{\hbar}|\langle 34|t|12\rangle|^2\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\,n_1n_2(1-n_3)(1-n_4)$$

$$\frac{2\pi}{\hbar}|\langle 12|t|34\rangle|^2\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\,n_3n_4(1-n_1)(1-n_2)$$

define $\frac{2\pi}{\hbar}|\langle 34|t|12\rangle|^2=\frac{1}{V^2}W(12;34)\,\delta_{p_1+p_2,p_3+p_4}\,\delta_{\sigma_1+\sigma_2=\sigma_3+\sigma_4}$

$$\Rightarrow I_{coll}[n_p]=\frac{1}{V^2}\sum_{p_2\sigma_2}\sum_{p_3\sigma_3}\sum_{p_4\sigma_4}W(12;34)\,\delta_{p_1+p_2,p_3+p_4}\,\delta_{\sigma_1+\sigma_2,\sigma_3+\sigma_4}$$

$$\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\left[n_3n_4(1-n_1)(1-n_2)-n_1n_2(1-n_3)(1-n_4)\right]$$

In most situations, we will use a relaxation time approximation

$$[I(n_p)]_{RT}=-\frac{\delta n}{\tau}$$

In the case of $\omega\tau\gg1$, we can neglect the collision integral.

$$\Rightarrow (s-\cos\theta)\,\delta n(p;q\omega)+\frac{\partial n^0}{\partial\varepsilon_p}\cos\theta\Big(\sum_{p'}f_{pp'}\,\delta n(p';q\omega)\Big)=0$$

we write $\delta n_p=-\frac{\partial n_p^0}{\partial\varepsilon_p}\nu_p$, ← the distortion of k_F in the direction of p

$$\Rightarrow (s-\cos\theta)\,\nu_p-\cos\theta\sum_{p'}f_{pp'}\Big(-\frac{\partial n_p^0}{\partial\varepsilon_p}\Big)\nu_{p'}=0$$

- expand $\nu_p=\sum_l Y_{l0}(\hat p)\,u_{l0}$

$$\Rightarrow \nu_p=\frac{\cos\theta}{s-\cos\theta}\cdot\sum_l\frac{1}{2l+1}F_l^s\,Y_{l0}(\hat p)\cdot u_{l0}=0$$

$$\sum_l Y_{l0}(\hat p)\,u_{l0}=F_l^s\sum_l\frac{1}{2l+1}\frac{\cos\theta}{s-\cos\theta}Y_{l0}(\hat p)\,u_{l0}$$

$$\Rightarrow \boxed{F_0^S \int \frac{d\Omega}{4\pi} \frac{\cos\theta}{S-\cos\theta} = 1}$$

only keep the zero-th order, we get the zero-sound

$\rightarrow$ the Solution :

$$S_0 = \begin{cases} 1 + 2e^{-2/1+F_0^S} & F_0^S \ll 1 \\ \sqrt{F_0^S/3} & F_0^S \gg 1 \end{cases}$$

§ spin-related transport

define $n_p(r,t) = \frac{1}{2} tr[n_{p,\alpha\beta}]$, $\vec{\sigma}_p(r,t) = \frac{1}{2} tr[n_{p,\alpha\beta} \vec{\tau}_{\beta\alpha}]$

$$\Rightarrow \boxed{n(p;rt) = n_p(r,t)\, \delta_{\alpha\alpha'} + \vec{\sigma}_p(r,t) \cdot \vec{\tau}_{\alpha\alpha'}}$$

(decompose into density and spin)

the quasi-particle energy can be written as

$$\boxed{\mathcal{E}(prt) = \mathcal{E}_p(r,t)\, \delta_{\alpha\alpha'} + \vec{h}_p(r,t) \cdot \vec{\tau}_{\alpha\alpha'}}\,.$$

Plug in the Boltzman Eq

$$\frac{\partial n(rpt)}{\partial t} + \frac{\partial}{\partial r}\left[\frac{\partial \mathcal{E}}{\partial p} \cdot n(rpt)\right] + \frac{\partial}{\partial p}\left[\frac{\partial \mathcal{E}}{\partial r} \cdot n(r,pt)\right]$$

$$\underbrace{-\frac{1}{i\hbar}[n(r,pt), \mathcal{E}(r,p,t)]}_{\text{Larmor procession}} = I_{\text{collision}}$$

Seperate variables

$$\boxed{\begin{aligned} &\frac{\partial n_p(rt)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial \mathcal{E}_p}{\partial p_i} n_p + \frac{\partial \vec{h}_p}{\partial p_i} \cdot \vec{\sigma}_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial \mathcal{E}}{\partial r_i} n_p - \frac{\partial \vec{h}_p}{\partial r_i} \cdot \vec{\sigma}_p\right] = I_{coll} \\ &\frac{\partial \vec{\sigma}_p(rt)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial \mathcal{E}_p}{\partial p_i} \vec{\sigma}_p + \frac{\partial \vec{h}_p}{\partial p_i} n_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial \mathcal{E}}{\partial r_i} \vec{\sigma}_p - \frac{\partial \vec{h}_p}{\partial r_i} \cdot n_p\right] = \frac{2}{\hbar} \vec{h}_p \times \vec{\sigma}_p + I_{coll.} \end{aligned}}$$

Lamor precession: in an external magnet field $\mathcal{H}$, it couples to electron spin as $-\frac{\gamma}{2}\mathcal{H}\cdot\sigma \Rightarrow \frac{\partial \vec{\sigma}_p}{\partial t} = \vec{\sigma}_p \times \gamma\mathcal{H} \Rightarrow \omega_0 = \gamma\mathcal{H}$

In the Fermi liquid, $h_p = -\gamma\frac{\hbar}{2}\mathcal{H} + 2\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\sigma_{p'}$

↳ contribution from interaction

in the uniform system, we have

$$\frac{\partial \vec{\sigma}_p}{\partial t} = \gamma\,\vec{\sigma}_p \times \vec{\mathcal{H}} - \frac{4}{\hbar}\int\frac{d^3p'}{(2\pi)^3}\,(\vec{\sigma}_p \times \vec{\sigma}_{p'})$$

define $\delta\,\sigma(\hat{\Omega}) = 2\int\frac{dp\cdot p^2}{(2\pi)^3}\sigma(p,\hat{\Omega})$, (we integrate out radius direction).

$$\Rightarrow 2\int\frac{p^2dp}{(2\pi)^3}\frac{\partial\vec{\sigma}}{\partial t}(p,\hat{\Omega}) = 2\int\frac{p^2dp}{(2\pi)^3}\,\vec{\sigma}(p,\hat{\Omega})\times(\gamma\vec{\mathcal{H}}) - 8\int\frac{d\Omega'}{4\pi}\int\frac{p'^2dp'}{(2\pi)^3}\int\frac{p^2dp}{(2\pi)^3}\,f^a(pp')\,\sigma_p\times\sigma_{p'}$$

$$\frac{\partial}{\partial t}\sigma(\hat{\Omega}_p) = \gamma\,\sigma(\hat{\Omega}_p)\times\vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega'_p}{4\pi}\,f^a(p,p')\,\hat{\sigma}(\Omega_p)\,\hat{\sigma}(\Omega_{p'})$$

In the external field $\sigma(\hat{\Omega}_p) = \sigma^0 + \delta\sigma(\hat{p}) \Rightarrow$

$$\frac{\partial}{\partial t}\delta\vec{\sigma}(\Omega_p) = \gamma\,\delta\vec{\sigma}(\Omega_p)\times\vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega'_p}{4\pi}\,f^a(pp')\left[\delta\sigma(\hat{p})\times\sigma^0 + \sigma^0\times\delta\sigma(\hat{p}')\right]$$

define $\delta\sigma_+(\hat{\Omega}_p) = \delta\sigma_x(\hat{\Omega}_p) + i\,\delta\sigma_y(\hat{\Omega}_p)$

$$\frac{\partial}{\partial t}\delta\sigma_+(\Omega_p) = -i\left[\omega_0\,\delta\sigma_+(\Omega_p) - \frac{2\sigma^0}{\hbar}\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\left(\delta\sigma_+(\Omega_p) - \delta\sigma_+(\Omega_{p'})\right)\right)$$

$$= -i\left[\left(\omega_0 - \frac{2}{\hbar}N^{-1}(0)\,F_0^a\,\sigma^0\right)\delta\sigma_+(\Omega_p) + \frac{2}{\hbar}\sigma^0\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\,\delta\sigma_+(\Omega_{p'})\right]$$

expand $\delta\sigma_+(\Omega_p) = \sum_{lm}\delta\sigma_+(lm)\,Y_{lm}(\Omega_p)$

$$\int \frac{d\Omega'}{4\pi} f^a(p,p')\, \delta\sigma_+(\Omega_{p'}) = N^{-1}(0) \int \frac{d\Omega_{p'}}{4\pi} F^a_l \frac{4\pi}{2l+1} Y_{lm}(\Omega_p) Y^*_{lm}(\Omega_{p'}) \sum_{l'm'} Y_{l'm'}(\Omega_{p'})\, \delta\sigma_+(l'm')$$

$$= N^{-1}(0) \sum_{l,m} \frac{F^a_l}{2l+1} Y_{lm}(\Omega_p)\, \delta\sigma_+(lm)$$

$$\Rightarrow \quad \frac{\partial}{\partial t} \delta\sigma_+(lm) = -i\left\{\omega_0 - \frac{2}{\hbar} N^{-1}(0) F^a_0 \sigma^0 + N^{-1}(0) \frac{F^a_l}{2l+1}\right\} \delta\sigma_+(lm)$$

$$\Rightarrow \quad \omega_{l+} = \left\{\omega_0 - \frac{2}{\hbar} \sigma_0 N^{-1}(0) \left[F^a_0 - \frac{F^a_l}{2l+1} \right]\right\}$$

$$\sigma^0 = \frac{\gamma\hbar}{2} \frac{N(0)}{1+F^a_0} \mathcal{H}, \quad \omega_0 = \gamma \mathcal{H}$$

$$\Rightarrow \quad \boxed{\frac{\omega_{l+}}{\omega_0} = \frac{1 + F^a_l/2l+1}{1+F^a_0}}$$

the $l=0$ channel Larmor frequency is not modified by interaction because Spin is conserved by interaction!

~~§ spin hydrodynamic equations~~

Integrate over momentum for the Boltzman transport equation $\Rightarrow$

$$\frac{\partial}{\partial t} \vec{\sigma}(r,t) + \frac{\partial}{\partial r_i} \vec{j}_i(\vec{r},t) = -\frac{2}{\hbar} \int \frac{d\vec{p}}{(2\pi)^3} \vec{\sigma}_p \times \left[-\frac{\gamma}{2} \mathcal{H}(r,t) + \int \frac{d\vec{p}'}{(2\pi)^3} f^a(p,p') \vec{\sigma}_{p'}\right)$$

$$\boxed{\frac{\partial}{\partial t} \vec{\sigma}(r,t) + \frac{\partial}{\partial r_i} \vec{j}_i(\vec{r},t) = \gamma\, \vec{\sigma}(r,t) \times \mathcal{H}(r,t)}$$ (inteaction part cancels).

where $\vec{\sigma}(r,t) = 2 \int \frac{d^3p}{(2\pi)^3} \vec{\sigma}(r,p,t)$

$$\vec{j}_i(r,t) = 2 \int \frac{d^3p}{(2\pi)^3} \left(\frac{\partial \varepsilon_p}{\partial p_i} \vec{\sigma}(r\,p\,t) + \frac{\partial \vec{h}_p}{\partial p_i} n_p(r,p,t) \right]$$

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\left[\frac{\partial\varepsilon_p^0}{\partial p_i}\,\vec{\sigma}(r,p,t) - \frac{\partial n_p^0}{\partial p}(r,p,t)\,\vec{h}_p\right] \quad \text{(linearize)}.$$

$$= 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial\varepsilon_p}\,\vec{h}_p\right)$$

$$= 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial\varepsilon_p}\left(-\frac{\gamma}{2}\frac{\hbar}{2}\mathcal{H} + 2\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}\right)\right]$$

$$= 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p - 4\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\frac{\partial n_p^0}{\partial\varepsilon_p}\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}$$

$$2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\frac{\partial n_p^0}{\partial\varepsilon_p}\, f^a(p,p') = -N(0)\int\frac{d\Omega}{4\pi}\sum_l f_l^a\, P_l(\cos\theta)\, v_F\cdot\cos\theta \quad \text{(set } p' \text{ along z axis)}.$$

$$= -\frac{F_1^a}{3}\, v_F$$

$$\Rightarrow \vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p + 2\int\frac{d^3p'}{(2\pi)^3}\,\frac{F_1^a}{3}\, v_{p_i'}\,\vec{\sigma}_{p'}$$

$$\boxed{\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p\left(1+\frac{F_1^a}{3}\right)}$$