Lect 9 Landau Fermi liquid (II)

Lect 7 Landau Fermi Liquid (I)

§ Adiabatic continuity

— Each state of the free fermi gas corresponds to a state of the interacting system by turning on interaction adiabatically.

At zero temperature, the quasi particle distribution satisfies

$$n^0_{p\sigma} = \begin{cases} 1 & k \le k_F \\ 0 & k > k_F \end{cases},$$

i.e. the very existence of Fermi surface.

We can creat excitations by removing some particles inside the Fermi surface to outside. And we define its energy as

$$E - E_0 = \sum_{p\sigma} \epsilon(p)\, \delta n_{p,\sigma} \quad \text{and so do}$$

$$\vec{P} = \sum_{p\sigma} \vec{p}\, \delta n_{p\sigma}, \quad \vec{S} = \sum_{\sigma} \vec{\sigma}\, \delta n_{p\sigma}.$$

— Let us expand $\varepsilon(k) = \left(\frac{d\varepsilon}{dk}\right)_{k_F} (k - k_F)$ i.e. $v_F = \left(\frac{d\varepsilon}{dk}\right)_{k_F}$ and $m^* = \frac{P_F}{v_F}$.

effective mass.

§ interaction between quasi-particles

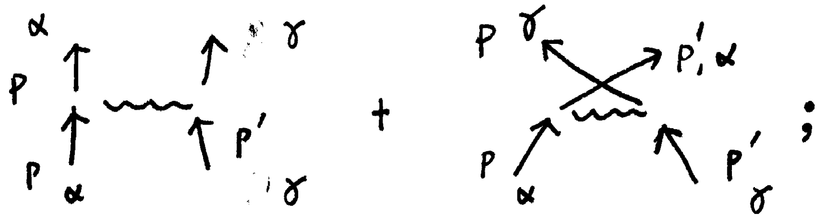
$$\delta E = \sum_{p\sigma} \underbrace{\frac{\delta E}{\delta n_{p\sigma}}}_{\varepsilon(p\sigma)} \delta n_{p\sigma} + \frac{1}{2} \sum_{pp',\sigma\sigma'} \underbrace{\frac{\delta^2 E}{\delta n_{p\sigma}\, \delta n_{p'\sigma'}}}_{\frac{1}{V} f(\vec{p}\vec{p}'; \sigma\sigma')} \delta n_{p\sigma}\, \delta n_{p'\sigma'}$$

unite of energy ⊗ volume

— More generally, we don't need to specify the spin quatization axis, but represent it as density-matrix $\delta n_{p,\alpha\beta}$

$$\delta E = \sum_{p\sigma} \varepsilon_{p,\alpha\beta}\, \delta n_{p,\beta\alpha} + \frac{1}{2} \sum_{\substack{pp',\alpha\beta \\ \gamma\delta}} f_{\alpha\beta;\gamma\delta}(\vec p\, \vec p\,')\, \delta n_{p,\beta\alpha}\, \delta n_{p',\delta\gamma}$$

At H-F level $\Rightarrow$ $$f_{\alpha\beta;\gamma\delta}(\vec p, \vec p\,') = V(0)\,\delta_{\alpha\beta}\cdot\delta_{\gamma\delta} - V(\vec p - \vec p\,')\,\delta_{\alpha\delta}\,\delta_{\gamma\beta}$$



using $\frac{1}{2}\left[\vec\sigma_{\alpha\beta}\cdot\vec\sigma_{\gamma\delta} + \delta_{\alpha\beta}\delta_{\gamma\delta}\right] = \delta_{\alpha\delta}\,\delta_{\beta\gamma}$

$$\Rightarrow \left| f_{\alpha\beta;\gamma\delta}(\vec p\, \vec p\,') = \left[V(0) - \frac{1}{2} V(p-p')\right] \delta_{\alpha\beta}\cdot\delta_{\gamma\delta} - \frac{1}{2} V(p-p')\, \vec\sigma_{\alpha\beta}\cdot\vec\sigma_{\gamma\delta}\right.$$

Generally, the interaction function can be represented as

$$\boxed{f_{\alpha\beta;\gamma\delta}(\vec p, \vec p\,') = f^{s}(\vec p\, \vec p\,')\,\delta_{\alpha\beta}\delta_{\gamma\delta} + f^{a}(\vec p\, \vec p\,')\,\vec\sigma_{\alpha\beta}\cdot\vec\sigma_{\gamma\delta}}$$

because of the symmetry of $SU(2)$.

Or we fix the quantization axis along z-axis.

$$f^{s} = (f_{\uparrow\uparrow} + f_{\uparrow\downarrow})/2 \; ; \quad f^{a} = (f_{\uparrow\uparrow} - f_{\uparrow\downarrow})/2 .$$

$f^{s,a}(\vec p\, \vec p\,')$ describes the forward scattering amplitude, which ~~is the~~ mark the fixed points of Fermi liquid in the language of RG.

§ Fermi liquid corrections to physical ~~???~~ quantities.

dimensionless Landau interaction function

$$f_{s,a}(\cos\theta) = \sum_l f_{l;\,s,a}\, P_l(\cos\theta)$$

$$F_{s,a} = N_0\, f_{l;s,a} \quad ; \quad N_0 \text{ density of state}$$

The interaction effects are summarized in the two sets of Landau parameters.

☆ S-wave channel : molecular method

spin-susceptibilies :

$$f_0^a\, \sigma\sigma' = N^{-1}(0)\, F_0^a\, \sigma\sigma'$$

$$\delta\mathcal{E}^{(2)} = \frac{1}{2} N^{-1}(0)\, F_0^a \sum_{pp'\sigma\sigma'} \sigma\sigma'\, \delta n_{p\sigma}\, \delta n_{p\sigma'} = \frac{1}{2} N^{-1}(0)\, F_0^a\, (S_z)^2$$

define molecule field $\quad E = -\int \vec{h}_{mol}\cdot d\vec{S}$

$$\Rightarrow h_{mol}(s) = -\frac{\delta\mathcal{E}}{\delta S_z} = -N^{-1}(0)\, S_z\, F_0^a$$

$$h_{tot} = h_{ex} + h_{mol} = h_{ex} - N^{-1}(0)\, S_z\, F_0^a$$

$$S_z = \chi_0\, h_{tot} = \chi_0\, h_{ext} - \chi_0 N^{-1}(0)\, S_z\, F_0^a$$

$$S_z(1 + \chi_0 F_0^a N^{-1}(0)) = \chi_0\, h_{ext} \Rightarrow \boxed{\chi = \frac{\chi_0}{1+\chi_0 F_0^a (N(0))^{-1}}}$$

Compressibility $\quad f_0^s \cancel{\;\;}' = N^{-1}(0)\, F_0^s$

$$\delta \mathcal{E}^{(2)} = \frac{N^{-1}(0)}{2} F_0^s \sum_p \delta n_p \delta n_{p'} = \frac{1}{2} (N(0))^{-1} F_0^s (\delta n)^2$$

$$h_{mol} = - N^{-1}(0)\, F_0^s\, \delta n \quad \Rightarrow \quad \boxed{\frac{dn}{d\mu} = \frac{N(0)}{1+F_0^s}}$$

☆ p-wave channel: effective mass.

define $\quad n(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, n_{p,\sigma}(r,t)$ $\qquad$ allow a slow spatial variation.

$$\vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3}\, \vec{\nabla}_p \mathcal{E}_{p\sigma}(r,t)\, n_{p\sigma}(r,t)$$

linearizing the expression of $\vec{j}(r,t)$, by using

$$\mathcal{E}_{p\sigma}(r,t) = \mathcal{E}_p^0 + \int \frac{d^3p'}{(2\pi)^3} f^s_{\sigma\sigma'}(p\,p')\, \delta n_{p'\sigma'}(r,t)$$

$$n_{p\sigma}(r,t) = n_p^0 + \delta n_{p,\sigma}(r,t)$$

$$\vec{j}(r,t) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \mathcal{E}^0_{p\sigma}\, \delta n_{p\sigma}(r\,t) + \nabla_p \delta \mathcal{E}_{p\sigma}(r\,t) . n_p^0$$

↙ partial derivative

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} \nabla_p \mathcal{E}_p^0\, \delta n_{p\sigma}(r,t) - \nabla_p n_p^0\, \delta \mathcal{E}_{p\sigma}(r,t)$$

$$= \sum_\sigma \int \frac{d^3p}{(2\pi)^3} v_p \left[\delta n_{p\sigma}(r,t) - \frac{\partial n^0_{p\sigma}}{\partial \mathcal{E}_p} \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p\,p')\, \delta n_{p'\sigma}(r\,t)\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} v_p\, \delta n_p(r,t) + \int \frac{d^3p}{(2\pi)^3} v_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \mathcal{E}_p}\right) \int \frac{d^3p'}{(2\pi)^3} f^s(p\,p')\, \delta n_{p'}(r\,t)$$

$$\int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(p,p') = N(0) \int \frac{d\Omega}{4\pi} \sum_l f^s_l P_l(\cos\theta)\, v_F \cos\theta\, \hat{z}$$

↳ set p' along z-axis in the p-space

$$= \frac{N(0)}{3} f_1^s \vec{v}_F \hat{z},$$

{ other two direction average to zero

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(-\frac{\partial n^0_{p\sigma}}{\partial \varepsilon_p}\right) f^s(\vec{p},\vec{p}\,') = \frac{N(0)}{3} f_1^s \vec{v}_{p'} = \frac{F_1^s}{3} \vec{v}_{p'}$$

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \,\delta n_p(r,t) + \frac{F_1^s}{3} \int \frac{d^3p'}{(2\pi)^3} \vec{v}_{p'} \,\delta n_{p'}(r,t)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{v}_p \left(1 + \frac{F_1^s}{3}\right) \delta n_p(r,t) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m^*} \left(1+\frac{F_1^s}{3}\right) \delta n_p(r,t)$$

on other hand, by adiabatic continuity

$$\vec{j}(r,t) = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{m} \,\delta n_p(r,t) \quad \Rightarrow \quad \boxed{\frac{1}{m} = \frac{1}{m^*}\left(1+\frac{F_1^s}{3}\right)}$$

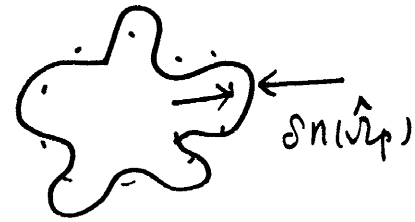
Similarly, we can derive spin current

$$j_i^\mu = 2 \int \frac{d^3p}{(2\pi)^3} \left(1+\frac{F_1^a}{3}\right) \frac{p_i}{m^*} \sigma^\mu_p(r,t)$$

we can define spin-effective mass $\frac{1}{m_s^*} = \frac{1}{m^*}\left(1+\frac{F_1^a}{3}\right)$

$$\boxed{\frac{m_s^*}{m} = \frac{1+\frac{1}{3}F_1^s}{1+\frac{1}{3}F_1^a}}$$

§ For general channels $F_l^{a,s}$



$$\delta n = V\int \frac{p^2 dp}{(2\pi)^3}\int d\Omega_p\, \delta n(p,\Omega_p) = V\int d\Omega\, \delta n(\hat{\Omega}_p)$$

where $\delta n(\hat{\Omega})$ is defined as $\int \frac{p^2 dp}{(2\pi)^3}\delta n(p,\Omega_p)$, i.e. integrate over radius direction.

we expand the angular distribution in terms of harmonic oscillators

$$\delta n(\hat{\Omega}_p) = \sum_{lm} \delta n_{lm} Y_{lm}(\hat{\Omega}_p)$$

$$E^{(2)} = \frac{1}{2V}\sum_{pp'} f_{\sigma\sigma'}(\hat{p}\,\hat{p}')\,\delta n_{p\sigma}\,\delta n_{p'\sigma'} = \frac{V}{2}\int d\Omega_p d\Omega_{p'}\, f_{\sigma\sigma'}(\hat{p}\,\hat{p}')\,\delta n_\sigma(\Omega_p)\,\delta n_{\sigma'}(\Omega_{p'})$$

$$= \frac{V}{2} N^{-1}(0)\int d\Omega_p d\Omega_{p'} \underline{\sum_{lm} F_l^s \frac{4\pi}{2l+1} Y^*_{lm}(\Omega_p)\, Y_{lm}(\Omega_{p'})} \leftarrow \text{addition theorem}$$

$$\Big[\Big(\sum_{l_1 m_1} Y_{l_1 m_1}(\Omega_p)\,\delta n^s_{l_1 m_1}\Big)\Big(\sum_{l_2 m_2} Y_{l_2 m_2}(\Omega_{p'})\,\delta n^s_{l_2 m_2}\Big) + (s\to a)\Big]$$

where $F_{\sigma\sigma'} = F^s + F^a \sigma\sigma'$, $\delta n_{s,a} = \delta n_\uparrow \pm \delta n_\downarrow$

$$E^{(2)} = \frac{V}{2} N^{-1}(0)\Big[\sum_{lm} F_l^s \frac{4\pi}{2l+1}\,\delta n^{*(s)}_{lm}\,\delta n^{(s)}_{lm} + (s\to a)\Big]$$

The kinetic energy increase

$$\delta E^{(1)} = \sum \varepsilon_p \delta n_p = V\int d\Omega \int \frac{p^2 dp}{(2\pi)^3}\varepsilon_p\, \delta n(p,\Omega_p)$$

$$\int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \delta n(p,\hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} v_F \cdot \frac{1}{2} (\delta p_F)^2 \leftarrow \quad \begin{array}{l} \varepsilon_p = v_F \cdot \delta p \\ p^2 \to p_F^2 \end{array}$$

Compare with $\int \frac{p^2 dp}{(2\pi)^3} \delta n(p,\hat{\Omega}_p) = \frac{p_F^2}{(2\pi)^3} \delta p_F = \delta n(\Omega_p)$

$$\Rightarrow \int \frac{p^2 dp}{(2\pi)^3} \varepsilon_p \delta n(p,\hat{\Omega}_p) = \frac{v_F}{2} [\delta n(\Omega_p)]^2 \Big/ \frac{p_F^2}{(2\pi)^3} = 4\pi N^{-1}(0) [\delta n_p(\Omega_p)]^2$$

$$\delta E^{(1)} = V N^{-1}(0) \int d\Omega [\delta n(\Omega_p)]^2 = 2\pi V N^{-1}(0) \sum_{lm} |\delta n^s_{lm}|^2 + |\delta n^a_{lm}|^2$$

$$\Rightarrow \quad \Delta E = 2V N^{-1}(0) \sum_{lm} \left\{ \left(1 + \frac{F_l^s}{2l+1}\right) |\delta n^s_{lm}|^2 + (s \to a) \right\}$$

From thermodynamic properties, we know

$$\Delta E = \sum_{lm} \frac{1}{2\chi^s_l} |\delta n^s_{lm}|^2 + (s \to a)$$

$$\Rightarrow \quad \frac{1}{\chi_{l\,FL}^{s,a}} = \frac{1}{\chi^{s,a}_{l,0}} \left(1 + \frac{F_l^{s,a}}{2l+1}\right)$$

i.e. $$\boxed{\chi^{s,a}_{FL,l} = \frac{\chi^{s,a}_{l,}}{1 + \frac{F^{s,a}}{2l+1}}}$$

in ^{3}He $\quad F_0^s \simeq 10.8. \quad F_a^0 \approx -0.75$

Compressibility is greatly reduced

spin-susceptibility is greatly enhanced!

§ density and spin dynamic response (RPA)

$$\chi_{density}(q,\omega) = \frac{\chi_0(q,\omega)}{1 + F_0^s N^{-1}(0)\, \chi_0(q,\omega)}$$

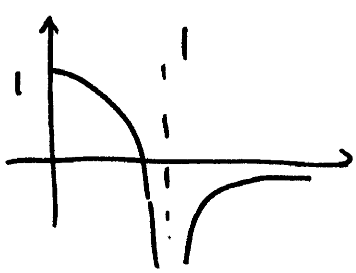
$$\chi_{spin}(q,\omega) = \frac{\chi_0(q,\omega)}{1 + F_0^a N^{-1}(0)\, \chi_0(q,\omega)}$$

$$\chi_0(q,\omega) = -\frac{2}{V} \sum_k \frac{n_{k+q} - n_k}{\omega + i0^+ - (\varepsilon_{k+q} - \varepsilon_k)} \qquad \text{Lindhard.}$$

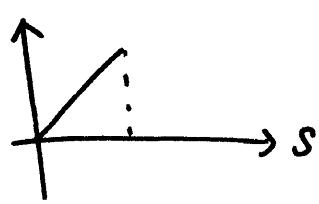
$$= N_0 f(s), \quad \text{where } f(s) = 1 - \frac{s}{2} \ln\left|\frac{1+s}{1-s}\right| + i\frac{\pi}{2} s\, \Theta(1-|s|)$$

$$s = v_F q/\omega$$

$\mathrm{Re}\, f(s)$



$\mathrm{Im}\, f(s)$



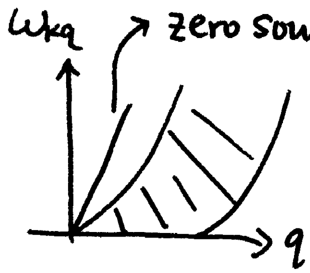
$$\Rightarrow \quad \chi_{density} = N_0 \frac{f(s)}{1 + F_0^s f(s)}$$

$$\chi_{spin} = N_0 \frac{f(s)}{1 + F_0^a f(s)}$$

Collective modes as poles

1. Zero sound : $\begin{cases} \mathrm{Im} f(s) = 0 \\ 1 + F_0^s f(s) = 0 \end{cases}$ at $s \gg 1$ $f(s) = \frac{-1}{3s^2}$

$$\Rightarrow \quad s^2 = \frac{F_0^s}{3} \qquad \text{i.e.} \quad \frac{\omega}{q} = \sqrt{\frac{F_0^s}{3}}\, v_F > v_F$$



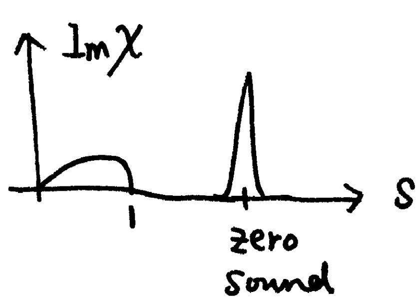
imaginary part of response function

$$\chi_{\text{density}}(q, \omega) = N(0) \frac{\mathrm{Re} f(s) + i\, \mathrm{Im} f(s)}{[1 + F_0^s \mathrm{Re} f(s)] + i F_0^s \mathrm{Im} f(s)}$$

$$= N(0) \frac{[\mathrm{Re} f(s)(1 + F_0^s \mathrm{Re} f(s)) + (\mathrm{Im} f(s))^2 F_0^s] - i\, \mathrm{Im} f(s)}{(1 + F_0^s \mathrm{Re} f(s))^2 + (F_0^s \mathrm{Im} f(s))^2}$$

at $s \ll 1$, $\mathrm{Im} f(s) \to \frac{s}{(1 + F_0^s)^2}$ is strongly suppressed.

$s > 1$, new delta peak at zero-sound frequency.



$(1 + F_0^a \mathrm{Re} f(s))^{-2} \gg 1$ as $s \to 0$

2. paramagnon model.

imaginary part is enhanced