Solution HW 1

§1. $\mathcal{L} = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega_0^2 x^2$

$$i\,G(x_b\ t_b ; x_a\ t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1} \prod_{i=1}^{N} \exp\Big[i \frac{m}{2} \frac{(x_i - x_{i-1})^2}{(\Delta t)^2} \Delta t - i \frac{m}{2} \omega_0^2 x_i^2 \Delta t \Big]$$

The classic solution to equation of motion reads

$$x_c(t) = \frac{1}{\sin\omega(t_a - t_b)} \left[x_a \sin\omega(t - t_b) - x_b \sin\omega(t - t_a) \right]$$

The classic solution:

$$\mathcal{L}_c = \frac{1}{2} m \left[\frac{1}{\sin\omega(t_a - t_b)}\right]^2 \Big[\omega^2 x_a^2 \cos^2\omega(t-t_b) + \omega^2 x_b^2 \cos^2\omega(t-t_a) - 2 x_a x_b \omega^2 \cos\omega(t-t_b) \cos\omega(t-t_a) \Big]$$

$$- \frac{1}{2} m \left[\frac{1}{\sin\omega(t_a - t_b)}\right]^2 \Big(\omega^2 x_a^2 \sin^2\omega(t-t_b) + \omega^2 x_b^2 \sin^2\omega(t-t_a) - 2 x_a x_b \omega^2 \sin\omega(t-t_b) \sin\omega(t-t_a) \Big]$$

$$= \frac{m}{2} \left[\frac{\omega}{\sin\omega(t_a - t_b)}\right]^2 \Big[x_a^2 \cos 2\omega(t-t_b) + x_b^2 \cos 2\omega(t-t_b) - 2 x_a x_b \cos \omega(2t - t_a - t_b) \Big]$$

$$\int_{t_a}^{t_b} \mathcal{L}\, dt = \frac{m}{2} \frac{\omega^2}{[\sin\omega(t_a - t_b)]^2} \Big(\frac{x_a^2}{2\omega} \sin 2\omega(t-t_b) \Big|_{t_a}^{t_b} + \frac{x_b^2}{2\omega} \sin 2\omega(t-t_b) \Big|_{t_a}^{t_b} - \frac{2 x_a x_b}{2\omega} \sin\omega(2t - t_a - t_b) \Big|_{t_a}^{t_b} \Big]$$

$$= \frac{m}{2} \frac{\omega^2}{\sin^2\omega(t_a - t_b)} \Big[- \frac{x_a^2}{2\omega} \sin 2\omega(t_a - t_b) - \frac{x_b^2}{2\omega} \sin 2\omega(t_a - t_b) - \frac{2 x_a x_b}{\omega} \sin 2\omega(t_b - t_a) \Big]$$

$$= \frac{m}{2} \frac{\omega}{\sin\omega(t_b - t_a)} \Big[\left(x_a^2 + x_a^2 \right) \cos\omega(t_b - t_a) + 2 x_a x_b \Big]$$

$$\Rightarrow \quad e^{iS_c} = \exp\left\{ \frac{im\omega}{2\sin\omega(t_b-t_a)} \left[(x_b^2 + x_a^2)\cos\omega(t_b-t_a) - 2x_a x_b \right] \right\}$$

Next let us calculate the Gaussian fluctuation

$$\delta x_j = x_j - x_c(t_j) \qquad \swarrow \delta x_0 = 0,\ \delta x_N = 0$$

$$\Rightarrow iG(x_b t_b; x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} e^{iS_c} \int dx_1 \cdots dx_{N-1} \prod_{i=1}^{N-1} \exp\left[i\frac{m}{2} \frac{(\delta x_j - \delta x_{j-1})^2}{\Delta t} - \frac{m}{2}\omega_0^2 \Delta t \delta x_i^2 \right]$$

$$\prod_{i=1}^{N} \exp\left[i\frac{m}{2} \frac{(\delta x_j - \delta x_{j-1})^2}{\Delta t} - i\frac{m}{2}\omega_0^2 \Delta t \delta x_i^2 \right]$$

$$= \exp\left(i \sum_{j,k=1,2,\cdots N-1} \delta x_j \, M_{jk} \, \delta x_k \right)$$

$$M_{jk} = \frac{m}{2\Delta t} \begin{bmatrix} 2-\omega_0^2(\Delta t)^2 & -1 & & \\ -1 & 2-(\omega_0\Delta t)^2 & & \\ & & \ddots & \\ & & -1 & 2-(\omega_0\Delta t)^2 \end{bmatrix}$$

The gaussian integral gives $\dfrac{\pi^{\frac{N-1}{2}}}{\left(\mathrm{Det}(-iM)\right)^{1/2}}$,

According to the formula in lecture notes.

$$\mathrm{Det}(-iM) = \left(\frac{m}{2\Delta t i}\right)^{N-1} \cdot \frac{\sinh(N u]}{\sinh u}, \quad \text{and} \quad 2\cosh u = 2 - \omega_0^2(\Delta t)^2$$

$$\Rightarrow i\,G(x_b t_b;\, x_a t_a) = \left(\frac{m}{2\pi i\Delta t}\right)^{N/2} \left(\frac{m}{2\Delta t i\pi}\right)^{-\left(\frac{N-1}{2}\right)} \left[\frac{\sinh(N u)}{\sinh u}\right]^{-1/2} e^{iS_{cl}}$$

$$= \left(\frac{m}{2\pi i\Delta t}\right)^{1/2} \left(\frac{\sinh N u}{\sinh u}\right)^{-1/2} e^{iS_{cl}}$$

$$\cosh u = 1 - \frac{1}{2}\omega_0^2(\Delta t)^2 \quad \Rightarrow \quad \cosh u = \cos \omega t \;\; \text{i.e.}\;\; u = i\omega\Delta t$$

$$\sinh u = \sin \omega\Delta t \qquad \sinh N u = \sin \omega(t_b - t_a)$$

$$\Rightarrow i\,G(x_b t_b;\, x_a t_a) = \left[\frac{m}{2\pi i\Delta t} \cdot \frac{\omega\Delta t}{\sin\omega(t_b - t_a)}\right]^{1/2} e^{iS_{cl}}$$

$$= \left[\frac{m\,\omega}{2\pi i \sin\omega(t_b - t_a)}\right]^{1/2} e^{iS_{cl}}$$

2. 1° We only need to sum over the Berry's phase on each site

$$\mathcal{Z} = \int D\hat{\Omega}_i(t\ \ ) \cdot \exp[+iS[\Omega_i(t,\ )]]$$

$$S[\hat{\Omega}] = \int_0^t dt' \left\{ S \sum \vec{A} \cdot \dot{\vec{\Omega}}_i(t,\ ) - H(\Omega_i(t,\ )) \right\}$$

where $\vec{A}(i,t)$ on each site represents the vector potential for a monopole

$$(\nabla \times \vec{A}) \cdot \hat{\Omega} = \varepsilon^{\alpha\beta\gamma} \frac{\partial A^\beta}{\partial \Omega^\alpha} \Omega^\gamma = 1,$$

$$H = \mp J \sum_{\langle ij \rangle} \vec{\Omega}(i) \cdot \vec{\Omega}(j)$$ for ferro / anti-ferro spin chain respectively.

2° The equation of motion for ferro-magnet.

$$\cdot \frac{\delta S}{\partial \delta \Omega} = S \sum_i \int_0^t dt' \ \hat{\Omega}_i \cdot (\dot{\hat{\Omega}}_i \times \delta\hat{\Omega}(t)) \quad - \frac{\partial H}{\partial \Omega} \delta\Omega \ ]$$

$$\Rightarrow S \dot{\hat{\Omega}}_i(t) = \hat{\Omega}_i(t) \times \left(- \frac{\partial H}{\partial \hat{\Omega}_i(t)} \right)$$

$$H = -JS^2 \sum_{\langle ij \rangle} \hat{\Omega}(i) \cdot \hat{\Omega}(j)$$

$$\Rightarrow \dot{\hat{\Omega}}_i(t) = JS \, \hat{\Omega}_i(t) \times (\hat{\Omega}_{i-1}(t) + \hat{\Omega}_{i+1}(t))$$

let us assume that $\Omega_i(t) = \hat{z} + \quad \Omega_x \hat{x} + \Omega_y \hat{y}$

$\Rightarrow$

$$\dot{\Omega}_x(i,t) = JS\left[\Omega_y(i,t)\left(\Omega_z(i-1,t)+\Omega_z(i+1,t)\right) - \Omega_z(i,t)\left(\Omega_y(i-1,t)+\Omega_y(i+1,t)\right)\right]$$

$$\dot{\Omega}_y(i,t) = JS\left[\Omega_z(i,t)\left(\Omega_x(i-1,t)+\Omega_x(i+1,t)\right) - \Omega_x(i,t)\left(\Omega_z(i-1,t)+\Omega_z(i+1,t)\right)\right]$$

→ Ferro-background ⇒

$$\dot{\Omega}_x(i,t) = JS\left[\Omega_y(i,t)\,2 - \left(\Omega_y(i-1,t)+\Omega_y(i+1,t)\right)\right]$$

$$\dot{\Omega}_y(i,t) = JS\left[\Omega_x(i-1,t)+\Omega_x(i+1,t) - 2\Omega_x(i,t)\right]$$

$$\dot{\Omega}_x(i,t) + i\dot{\Omega}_y(i,t) = iJS\left[\Omega_x(i-1,t)+i\Omega_y(i-1,t) + \Omega_x(i+1,t)+i\Omega_y(i+1,t) - 2\left(\Omega_x(i,t)+i\Omega_y(i,t)\right)\right]$$

$$\Rightarrow \dot{\Omega}_+(i,t) = iJS\left[\Omega_+(i-1,t)+\Omega_+(i+1,t) - 2\Omega_+(i,t)\right]$$

assuming $\Omega_+(i,t) = e^{ikR_i - i\omega t}$ $\Rightarrow$ $-i\omega_k = iJS\left[e^{ik}+e^{-ik}-2\right]$

$$\frac{\omega_k}{JS} = 2(1-\cos k) \rightarrow k^2$$

(as $k \to 0$).

3) For antiferro-magnet

$$\dot{\Omega}_x(i,t) = -JS\left[\Omega_y(i,t)\left[\Omega_z(i-1,t)+\Omega_z(i+1,t)\right] - \Omega_z(i,t)\left(\Omega_y(i-1,t)+\Omega_y(i+1,t)\right)\right]$$

$$\dot{\Omega}_y(i,t) = -JS\left[\Omega_z(i,t)\left[\Omega_x(i-1,t)+\Omega_x(i+1,t)\right] - \Omega_x(i,t)\left(\Omega_z(i-1,t)+\Omega_z(i+1,t)\right)\right]$$

plug in $\Omega_{2i} = \hat{z} + n_{x,2i}\,\hat{x} + n_{y,2i}\,\hat{y}$

$$\Omega_{2i+1} = -\hat{z} - n_{x,2i+1}\,\hat{x} - n_{y,2i+1}\,\hat{y}$$

$$\dot{n}_x(2i,t) = -JS\left[-2n_y(2i,t) + n_y(2i-1,t) + n_y(2i+1,t)\right]$$

$$-\dot{n}_x(2i+1,t) = -JS\left[-2n_y(2i+1,t) + n_y(2i,t) + n_y(2i+2,t)\right]$$

$$\dot{n}_y(2i,t) = -JS\,[-\,n_x(2i-1,t) - n_x(2i+1,t) + 2n_x(2i,t)]$$

$$-\dot{n}_y(2i+1,t) = -JS\,[-\,n_x(2i,t) - n_x(2i+2,t) + 2n_x(2i+1,t)]$$

$\Rightarrow$

$$\dot{n}_x(2i) + i\dot{n}_y(2i) = -iJS\,[2(n_x(2i) + in_y(2i)) - (n_x(2i-1) + in_y(2i-1)) - (n_x(2i+1) + in_y(2i+1))]$$

$$-\dot{n}_x(2i+1) - i\dot{n}_y(2i+1) = -iJS\,[2(n_x(2i+1) + in_y(2i+1)) - (n_x(2i) + in_y(2i)) - (n_x(2i+2) + in_y(2i+2))]$$

i.e. $$\dot{n}_+(2i) = -iJS\,[2n_+(2i) - n_-(2i-1) - n_-(2i+1)]$$

$$\dot{n}_+(2i+1) = iJS\,[2n_+(2i+1) - n_-(2i) - n_-(2i+2)]$$

try $$\begin{bmatrix} n_+(2i) \\ n_+(2i+1) \end{bmatrix} = \begin{bmatrix} e^{ikR_{2i}}\,a \\ e^{ikR_{2i+1}}\,b \end{bmatrix} e^{-i\omega t}$$

$$\Rightarrow \quad \begin{aligned} -i\omega a &= -iJS\,[2a - (e^{-ik} + e^{ik})b] \\ -i\omega b &= iJS\,[2b - (e^{-ik} + e^{ik})a] \end{aligned}$$

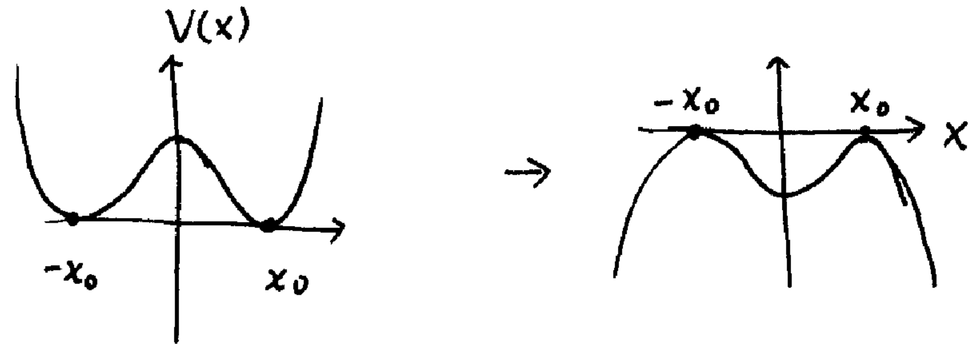
i.e. $$\begin{aligned} \omega a &= 2JS\,[a - b\cos k] \\ \omega b &= -2JS\,[b - a\cos k] \end{aligned} \quad \Rightarrow \quad \begin{aligned} (\omega - 2JS)a + 2bJS\cos k &= 0 \\ 2bJS\cos k\,a + (\omega + 2JS)a &= 0 \end{aligned}$$

$$\Rightarrow \begin{vmatrix} \omega - 2JS & 2JS\cos k \\ 2JS\cos k & \omega + 2JS \end{vmatrix} = 0 \Rightarrow \omega^2 - (2JS)^2 = (2JS\cos k)^2$$

$$\Rightarrow \omega^2 = (2JS)^2\,(1 - \cos^2 k)$$

$$\Rightarrow \omega = 2JS\sqrt{1 - \cos^2 k} \xrightarrow[k\to 0]{} 2JS\,|k|$$

3.



1. classical path for $S = +\int_{-T/2}^{+T/2} d\tau \; \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)$

$\Rightarrow m\frac{d^2x}{d\tau^2} = \frac{\partial V(x)}{\partial x}$ consider the solution $t \to -\infty \quad x = -x_0$, $t \to +\infty \quad x = x_0$

we have $\frac{dx}{d\tau} = \sqrt{\frac{2V(x)}{m}}$

$$S_0 = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \; \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \; \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 \times 2 = \frac{m}{\hbar}\int_{-x_0}^{x_0} dx \, \frac{dx}{d\tau}$$

$$= \frac{m}{\hbar}\int_{-x_0}^{x_0} dx \sqrt{\frac{2V(x)}{m}} = \int_{-x_0}^{x_0} dx \sqrt{\frac{2mV(x)}{\hbar^2}} = \int_{-x_0}^{x_0} \frac{dx}{\hbar} \sqrt{2m} \; \frac{\sqrt{g}}{2} \left|(x^2 - x_0^2)\right|$$

$$= \sqrt{\frac{mg}{2}} \; \frac{1}{\hbar} \int_{-x_0}^{x_0} (x_0^2 - x^2)\, dx = \sqrt{\frac{mg}{2}} \; \frac{1}{\hbar} \; \frac{4x_0^3}{3} = \frac{\sqrt{mg x_0^4}}{\hbar} \cdot x_0 \frac{2\sqrt{2}}{3}$$

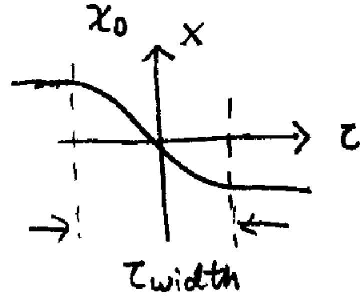
$$\frac{dx}{d\tau} = \sqrt{\frac{2V(x)}{m}} = \sqrt{\frac{g}{2m}} \; (x^2 - x_0^2) \Rightarrow \frac{dx}{x^2 - x_0^2} = \sqrt{\frac{g}{2m}} \; d\tau$$

$$\frac{dx}{2x_0}\left[\frac{-1}{x+x_0} + \frac{1}{x-x_0}\right] = \sqrt{\frac{g}{2m}} \, d\tau \Rightarrow \frac{1}{2x_0} \ln\frac{x-x_0}{x+x_0} = \sqrt{\frac{g}{2m}} \, d\tau$$

$$\frac{x-x_0}{x+x_0} = c \cdot e^{2x_0\sqrt{\frac{g}{2m}}\,\tau}$$, set $\tau = 0$, $x = 0$

$$\Rightarrow \frac{x - x_0}{x + x_0} = - e^{2x_0\sqrt{\frac{g}{2m}}\tau} \Rightarrow \frac{x}{x_0} = \frac{1 - e^{2x_0\sqrt{\frac{g}{2m}}\tau}}{1 + e^{2x_0\sqrt{\frac{g}{2m}}\tau}}$$

$$\Rightarrow \frac{x}{x_0} = -\tanh\left(x_0\sqrt{\frac{g}{2m}}\,\tau\right) \quad \text{i.e.} \quad \boxed{\tau_{width} \simeq \left(x_0\sqrt{\frac{g}{2m}}\right)^{-1}}$$



2. The action of n-~~instantin~~ events is

$$e^{-S_n} = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \cdot \frac{(K e^{-S_0/\hbar} T)^n}{n!}$$

$\Rightarrow$ the averge number of in~~stantin~~

$$\langle n \rangle = \sum_{n=0}^{\infty} \frac{n (K e^{-S_0/\hbar} T)^n}{n!} \Big/ \sum_{n=0}^{\infty} \frac{(K e^{-S_0/\hbar} T)^n}{n!}$$

$$= K e^{-S_0/\hbar} T \cdot \sum_{n=1}^{\infty} \frac{(K e^{-S_0/\hbar} T)^{n-1}}{(n-1)!} \Big/ \sum_{n=0}^{\infty} \frac{(K e^{-S_0/\hbar} T)^n}{n!}$$

$$= K e^{-S_0/\hbar} T$$

$\Rightarrow$ the instantanm density $\dfrac{\langle n \rangle}{T} = K e^{-S_0/\hbar}$

$$\sim \sqrt{\frac{g x_0^2}{m}}\; e^{-S_0/\hbar}$$

The dilute limit is valid if $\dfrac{\langle n \rangle}{T} \cdot \tau_{chac} \ll 1$ (assumption)

i.e. $\sqrt{\dfrac{g x_0^2}{m}}\, e^{-S_0/\hbar} \cdot x_0^{-1}\sqrt{\dfrac{g}{2m}} = \boxed{e^{-S_0/\hbar} \ll 1}$