Lect 2 Monopole harmonics

$$H = \frac{(P - \frac{e}{c}A)^2}{2m} \quad \text{where} \quad \vec{A} = \frac{g}{r}\frac{\vec{n}\times\vec{r}}{r + (\vec{r}\cdot\hat{n})} = \frac{g}{r}\frac{1-\cos\theta}{\sin\theta}\,\hat{e}_\varphi$$

check that $\nabla\times\vec{A} = \frac{g}{r^2}\,\hat{e}_r$.

Use the formula $\nabla\times\vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{e}_r$

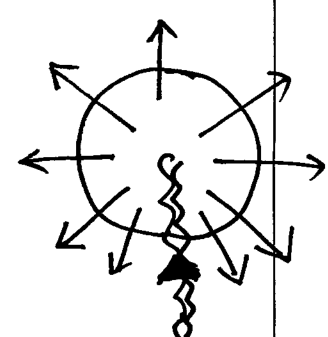
$$+\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)\right]\hat{e}_\theta + \left[\frac{1}{r}\frac{\partial}{\partial r}[rA_\theta] - \frac{1}{r}\frac{\partial}{\partial\theta}A_r\right]\hat{e}_\varphi$$

$$\Rightarrow \quad \nabla\times\vec{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{g}{r}(1-\cos\theta)\right) = \frac{g}{r^2}\,\hat{e}_r.$$

Dirac string: singularity at the south pole.

$\oint\vec{A}\cdot d\vec{l} = -4\pi g$

for infinite-simal loop around south pole.



Electron goes around the Dirac string and then picks up a phase $\frac{eg}{\hbar c}\cdot 4\pi$. If such a phase is $2n\pi$, then this string is invisible $\Rightarrow$ $\frac{eg}{\hbar c}4\pi = 2n\pi \Rightarrow \boxed{\frac{eg}{c} = \frac{n}{2}\hbar}$

Charge quantization

From classic. elern - manopole system, we learned that $\frac{eg}{c}$ is the minimum angular momentum of such a system, its minimum value is $\hbar/2$ according to quantum mechanics.

Define mechanical angular momentum $\vec{\Lambda} = \vec{r} \times (\vec{P} - \frac{e}{c}\vec{A})$.

$\vec{\Lambda}$ does not obey the commutation relation of angular mometum.

Please explicitly check that $H = \frac{(P - \frac{e}{c}A)^2}{2m}$ can be expressed as

$$H = \frac{-\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\cdots\right) + \frac{\hbar^2 \vec{\Lambda}^2}{2mr^2}\right].$$

(I leave it as a homework problem).

Hower, the spectra of the angular part $\overbrace{\text{are}}^{\hbar^2\vec{\Lambda}^2}$ no-longer $l(l+1)\hbar^2$.

we define $\vec{L} = \vec{r} \times (\vec{P} - \frac{e}{c}\vec{A}) - \frac{eg}{c}\hat{r}$ $\vec{L}$ satisfies the commutation relation of angular momentum, i.e.

($\hat{r}$ is the unit vector of $\vec{r}/r$)

$$\boxed{[L_i, L_j] = i\varepsilon_{ijk} L_k}$$

(I leave it as another home work problem).

we also have the following identities $\boxed{\vec{\Lambda}\cdot\hat{r} = \hat{r}\cdot\vec{\Lambda} = 0.}$

(Please check as an exercise!)

Then we have

$$\Lambda^2 = \left[\vec{L} + \frac{eg}{c}\hat{r}\right]^2 = L^2 + \left(\frac{eg}{c}\right)^2 + \frac{eg}{c}\left(\vec{L}\cdot\hat{r} + \hat{r}\cdot\vec{L}\right)$$

$$\vec{L}\cdot\hat{r} = \left[\vec{\Lambda} - \frac{eg}{c}\hat{r}\right]\cdot\hat{r} = -\frac{eg}{c}, \quad \hat{r}\cdot\vec{L} = -\frac{eg}{c}$$

$$\Rightarrow \vec{\Lambda}^2 = \vec{L}^2 - \left(\frac{eg}{c}\right)^2,$$ set $\frac{eg}{c} = \hbar q$, we have

(q can be half or integers)

$$\boxed{H = -\frac{\hbar^2}{2mr^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\cdots\right)\right] + \frac{\hbar^2}{2mr^2}\left[\vec{L}^2 - \hbar^2 q^2\right]}$$

By a litte algebra, and use the expressim in the spherical coordinate

$$\vec{p} = -i\hbar \left[\hat{e}_r \frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\hat{e}_\varphi}{r\sin\theta} \frac{\partial}{\partial \varphi} \right]$$

$$\Rightarrow \boxed{\vec{L} = \vec{r} \times (\vec{p} - \frac{e}{c}\vec{A}) - \hbar q \hat{r} = \frac{\hbar}{\sin\theta}\left[i\frac{\partial}{\partial\varphi} + q(1-\cos\theta) \right] \hat{e}_\theta - i\hbar \frac{\partial}{\partial\theta} \hat{e}_\varphi - \hbar q \hat{e}_r}$$

Change to Cartisan coordinates, by using

$$\hat{e}_r = \sin\theta\cos\varphi\, \hat{e}_x + \sin\theta\sin\varphi\, \hat{e}_y + \cos\theta\, \hat{e}_z$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\, \hat{e}_x + \cos\theta\sin\varphi\, \hat{e}_y - \sin\theta\, \hat{e}_z$$

$$\hat{e}_\varphi = -\sin\varphi\, \hat{e}_x + \cos\varphi\, \hat{e}_y$$

we have

$$\boxed{\begin{aligned} L_+ &= L_x + iL_y = \hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\,\mathrm{ctg}\,\theta \frac{\partial}{\partial\varphi} - q\frac{1-\cos\theta}{\sin\theta} \right] \\ L_- &= L_x - iL_y = \hbar e^{-i\varphi}\left[-\frac{\partial}{\partial\theta} + i\,\mathrm{ctg}\,\theta \frac{\partial}{\partial\theta} - q\frac{1-\cos\theta}{\sin\theta} \right] \\ L_z &= -i\hbar \frac{\partial}{\partial\varphi} - \hbar q \end{aligned}}$$

Hw problem: please derive these formulas in the boxes.

Also by a little algebra, we have

$$\boxed{\frac{L^2}{\hbar^2} = \frac{-1}{\sin\theta}\left[\sin\theta \frac{\partial}{\partial\theta}(\sin\theta \frac{\partial}{\partial\theta} \quad \right] + \frac{1}{\sin^2\theta}\left[i\frac{\partial}{\partial\varphi} + q(1-\cos\theta) \right]^2 + q^2}$$

Seek eigenstates $Y_{q,jm}(\theta,\varphi)$ satisfying

$$\boxed{\begin{aligned} L^2\, Y_{q,jm}(\theta,\varphi) &= j(j+1)\hbar^2\, Y_{q,jm}(\theta,\varphi) \\ L_z\, Y_{q,jm}(\theta,\varphi) &= m\hbar\, Y_{q,jm}(\theta,\varphi), \end{aligned}}$$

monopole harmonics

where $j = |q|, |q|+1, |q|+2, \cdots\cdots$

☆ Now we need to use our knowledge of D-matrix, which is also the wave-functions of rotating tops. We will build up the connetion between monopole harmonics and D-matrics.

For lecture 1, we know that the top wavefuction $\psi^{top}_{J;mk}(\alpha\beta\gamma) = \sqrt{\frac{2j+1}{8\pi^2}}\, D^{*j}_{mk}(\alpha\beta\gamma)$ which is the eigenstates for the angular momentum operators $L^2_{top}(\alpha\beta\gamma)$ and $L_{z,top}(\alpha\beta\gamma)$. We will see how to identify L^2_{top}, $L_{z,top}$ and ψ^{top} with the monopole harmonics $Y_{q,jm}(\theta,\varphi)$. Apparently, a major difference is that top has three Eulerian angles, while monopole harmonics has two angular variables.

Let us start with $\left[D^j_{m\,-q}(\alpha\beta\gamma)\right]^* = e^{im\alpha - iq\gamma}\, d^j_{m\,-q}(\beta)$

and we know that it satisfies

$$L_{z,top}\left[D^j_{m-q}(\alpha\beta\gamma)\right]^* = m\hbar\left[D^j_{m-q}(\alpha\beta\gamma)\right]^*$$

or $-i\hbar\frac{\partial}{\partial\alpha}\left[e^{im\alpha - iq\gamma}\, d^j_{m-q}(\beta)\right] = m\hbar\left[e^{im\alpha-iq\gamma}\, d^j_{m-q}(\beta)\right]$

but if we at the beginning set $\gamma = -\alpha$ before taking $\frac{\partial}{\partial\alpha}$, we have

$$-i\hbar \frac{\partial}{\partial\alpha} \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right] = (m+q)\hbar \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right]$$

$$\Rightarrow \boxed{\left(-i\hbar\frac{\partial}{\partial\alpha} - q\hbar\right) \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right] = m\hbar \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right]}$$

Again for top's $L_{top,+} = L_{top,x} + i L_{top,y}$

$$= i\hbar \left[+ e^{i\alpha} \cot\beta \frac{\partial}{\partial\alpha} - i e^{+i\alpha} \frac{\partial}{\partial\beta} - \frac{e^{i\alpha}}{\sin\beta} \frac{\partial}{\partial\gamma} \right]$$

$$L_{top,+} \left[D^{j}_{m-q}(\alpha\beta\gamma) \right]^* = \sqrt{(j-m)(j+m+1)} \left[D^{j}_{m+1,-q}(\alpha,\beta,\gamma) \right]^*$$

$$i\hbar \{ e^{i\alpha} [\cot\beta \frac{\partial}{\partial\alpha} \quad i\frac{\partial}{\partial\beta} - \frac{1}{\sin\beta} \frac{\partial}{\partial\gamma}] \left[e^{i(m\alpha - q\gamma)} d^{j}_{m-q}(\beta) \right]$$

$$= \sqrt{(j-m)(j+m+1)} \left[e^{i[(m+1)\alpha - q\gamma]} d^{j}_{m+1,-q}(\beta) \right]$$

$$\Rightarrow \hbar \left[-\cot\beta\, m + \frac{\partial}{\partial\beta} - \frac{q}{\sin\beta} \right] d^{j}_{m-q}(\beta) = \sqrt{(j-m)(j+m+1)}\, d^{j}_{m-q}(\beta)$$

$$\hbar \left[-\cot\beta\, (m+q) + \frac{\partial}{\partial\beta} - \frac{q}{\sin\beta}(1-\cos\beta) \right] d^{j}_{m-q}(\beta) = \sqrt{(j-m)(j+m+1)}\, d^{j}_{m-q}(\beta)$$

$$\Rightarrow \boxed{\begin{aligned} &\hbar e^{i\alpha} \left[i\cot\beta \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta} - q\frac{1-\cos\beta}{\sin\beta} \right] \left[e^{i(m+q)\alpha} d^{j}_{m-q}(\beta) \right] \\ &= \sqrt{(j-m)(j+m+1)} \left[d^{j}_{m+1,-q}(\beta)\, e^{i(m+q+1)\alpha} \right] \end{aligned}}$$

thus by seting $\gamma = -\alpha$, the D-matrix $\left[D^j_{m,-q}(\alpha\,\beta\,\gamma)\right]^*$

and identify $\theta = \beta$, $\varphi = \alpha$ $\quad \rightarrow \left[D^j_{m,-q}(\varphi,\theta,-\varphi)\right]^*$

$$\Rightarrow \boxed{\begin{aligned} &\hbar e^{i\varphi}\left[\frac{\partial}{\partial\theta} + i\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi} - q\,\frac{1-\cos\theta}{\sin\theta}\right]\left[D^j_{m,-q}(\varphi,\theta,-\varphi)\right]^* \\ &\qquad = \sqrt{(j-m)(j+m+1)}\,\left[D^j_{m+1,-q}(\varphi,\theta,-\varphi)\right]^* \\ &\left[-i\hbar\frac{\partial}{\partial\varphi} - \hbar q\right]\left[D^j_{m,-q}(\varphi,\theta,-\varphi)\right]^* = m\hbar\, \bar{D}^j_{m-q}(\varphi,\theta,-\varphi)\end{aligned}}$$

we anclude

$$\boxed{Y_{q,jm} = \sqrt{\frac{2j+1}{4\pi}}\left[D^j_{m-q}(\varphi,\theta,-\varphi)\right]^*}$$

please pay attention to the normalization factor, prove it!