①

# Solution to the anisotropic top. 99

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1. First prove for anisotropic top ($I_1 \neq I_2 \neq I_3$)

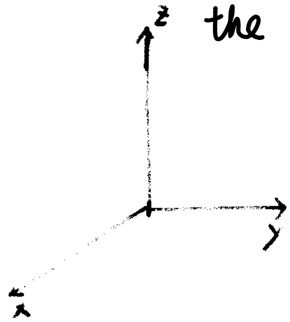
$[\hat{H}, \hat{L}_i] = 0$ , ($i = x, y, z$)

With top $H = \frac{\hat{J}_1^2}{2I_1} + \frac{\hat{J}_2^2}{2I_2} + \frac{\hat{J}_3^2}{2I_3}$

$\hat{J}_1, \hat{J}_2, \hat{J}_3$ are angular momentums in body frame

Compute $\hat{J}_1, \hat{J}_2, \hat{J}_3$ $\underbrace{\hat{L}_z}_{\hat{L}_x, \hat{L}_y,}$ with Euler angles

① Let $x, y, z$ be the axies

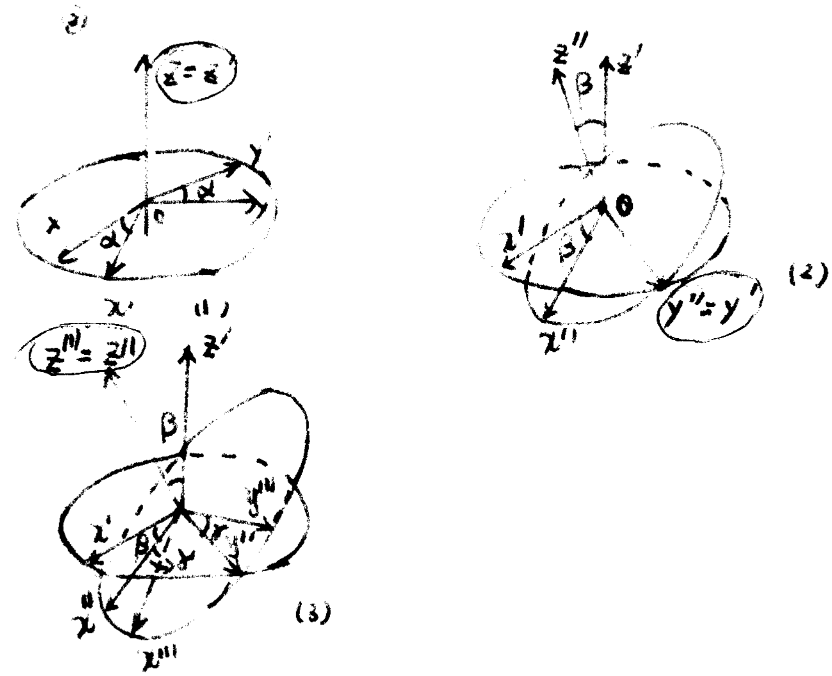



the projection of $\vec{L}$ onto $\hat{z}$ is $\hat{l}_z$,

we have $\hat{l}_z = -i\hbar \frac{\partial}{\partial \varphi}$ ;

Then for any axes $\hat{m}$, $\vec{L}$'s projection $\hat{l}_m = -i\hbar \frac{\partial}{\partial \varphi}$

($\varphi$ is the angle rotated about)




~~x, y, z~~ are the lab frame

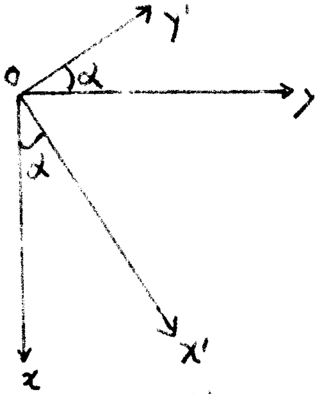
$x', y', z'$ rotate with the body

$x'', y'', z'', x''', y''', z''$

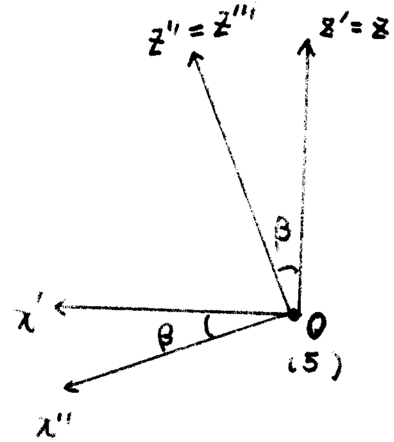
From (1), (3) $\hat{L}_z = -i\hbar \frac{\partial}{\partial \alpha}$

$\hat{J}_3 = -i\hbar \frac{\partial}{\partial \gamma}$

From (2). $\hat{L}_{y'} = -i\hbar\frac{\partial}{\partial\beta}$ .




(4) ,

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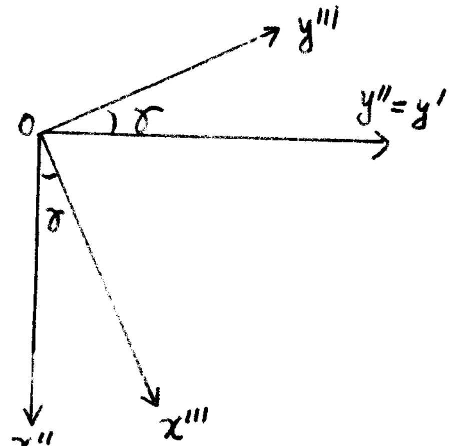


(6)

From (4) $\quad \hat{L}_{y'} = (\cos\alpha\,\hat{L}_y - \sin\alpha\,\hat{L}_x) = -i\hbar\frac{\partial}{\partial\beta}$ ①

From (5) $\quad \hat{L}_{z''} = \hat{J}_3 = -i\hbar\frac{\partial}{\partial\gamma} = \cos\beta\,\hat{L}_z + \sin\beta\,\hat{L}_{x'}$

From (4) $\quad \hat{L}_{x'} = \cancel{\hat{L}_x\cos\alpha + \hat{L}_y}\sin\alpha = \cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y$

$\therefore -i\hbar\frac{\partial}{\partial\gamma} = \cos\beta\,\hat{L}_z + \sin\beta(\cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y)$ ②

$\therefore \quad \cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y = -\frac{i\hbar}{\sin\beta}\frac{\partial}{\partial\gamma} + ctg\beta\, i\hbar\frac{\partial}{\partial\alpha}$ ③

Solve ①, ③ $\quad \hat{L}_x = -i\hbar\left[-\cos\alpha\, ctg\beta\frac{\partial}{\partial\alpha} - \sin\alpha\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]$ ④

$\hat{L}_y = -i\hbar\left[-\sin\alpha\, ctg\beta\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]$ ⑤

From (6) $L_x$ $\quad \hat{J}_1 = \hat{L}_{x'''} = \hat{L}_{x''}\cos\gamma + \hat{L}_{y'}\cdot\sin\gamma$

From (5) $\quad \hat{L}_{x''} = \cancel{\hat{L}_{x'}\cos\beta}\ \cos\beta\,\hat{L}_{x'} - \sin\beta\,\hat{L}_z$

From ③ $\quad \hat{L}_{x'} = \cos\alpha\,\hat{L}_x + \sin\alpha\,\hat{L}_y = -i\hbar\left[\frac{1}{\sin\beta}\frac{\partial}{\partial\gamma} + \cancel{\sin\alpha}\, ctg\beta\frac{\partial}{\partial\alpha}\right]$

$\therefore \hat{J}_1 = -i\hbar\cos\gamma\left[ctg\beta\frac{\partial}{\partial\gamma} + \frac{\cos^2\beta}{\sin\beta}\frac{\partial}{\partial\alpha}\right] - \overset{\cos\gamma}{\sin\beta}(-i\hbar)\frac{\partial}{\partial\alpha} + (-i\hbar)\sin\gamma\frac{\partial}{\partial\beta}$

$= -i\hbar\left[\sin\gamma\frac{\partial}{\partial\beta} - \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + ctg\beta\cos\gamma\frac{\partial}{\partial\gamma}\right]$ ⑥

$\hat{J}_2 = \hat{L}_{y'''} = -\sin\gamma\,\hat{L}_{x''} + \cos\gamma\,\hat{L}_{y'}$

$= -\sin\gamma(-i\hbar)\left[ctg\beta\frac{\partial}{\partial\gamma} + \frac{\cos^2\beta}{\sin\beta}\frac{\partial}{\partial\alpha}\right] + \sin\gamma\sin\beta(-i\hbar)\frac{\partial}{\partial\alpha}$

$+ \cos\gamma\cdot(-i\hbar)\frac{\partial}{\partial\beta}$

$$= -i\hbar\left(\cos\gamma\frac{\partial}{\partial\beta} + \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} - \operatorname{ctg}\beta\sin\gamma\frac{\partial}{\partial\gamma}\right) \qquad ⑦$$

$\therefore$

$$\hat{L}_x = -i\hbar\left[-\cos\alpha\operatorname{ctg}\beta\frac{\partial}{\partial\alpha} - \sin\alpha\frac{\partial}{\partial\beta} + \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]$$

$$\hat{L}_y = -i\hbar\left[-\sin\alpha\operatorname{ctg}\beta\frac{\partial}{\partial\alpha} + \cos\alpha\frac{\partial}{\partial\beta} + \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\alpha}$$

$$\hat{J}_1 = -i\hbar\left[\sin\gamma\frac{\partial}{\partial\beta} - \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + \operatorname{ctg}\beta\cos\gamma\frac{\partial}{\partial\gamma}\right]$$

$$\hat{J}_2 = -i\hbar\left[\cos\gamma\frac{\partial}{\partial\beta} + \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} - \operatorname{ctg}\beta\sin\gamma\frac{\partial}{\partial\gamma}\right]$$

$$\hat{J}_3 = -i\hbar\frac{\partial}{\partial\gamma}$$

③ First prove $[\hat{H}, L_{x,y,z}] = 0$, for $I_1 = I_2 \neq I_3$

$$\therefore \hat{H} = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3} = \frac{1}{2I_1}(J_1^2 + J_2^2 + J_3^2) + \left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)J_3^2$$

$\because J_1^2 + J_2^2 + J_3^2 = L^2$, 又 $[L^2, L_{x,y,z}] = 0$

$$\therefore [\hat{H}, L_{x,y,z}] = \left(\frac{1}{2I_3} - \frac{1}{2I_1}\right)[J_3^2, L_{x,y,z}]$$

$$\because [J_3, L_x] = -\hbar^2\left[\left[\frac{\partial}{\partial\gamma}, -\cos\alpha\operatorname{ctg}\beta\frac{\partial}{\partial\alpha}\right] + \left[\frac{\partial}{\partial\gamma}, -\sin\alpha\frac{\partial}{\partial\beta}\right] + \left[\frac{\partial}{\partial\gamma}, \frac{\cos\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]\right] = 0$$

$$[J_3, L_y] = -\hbar^2\left(\left[\frac{\partial}{\partial\gamma}, -\sin\alpha\operatorname{ctg}\beta\frac{\partial}{\partial\alpha}\right] + \left[\frac{\partial}{\partial\gamma}, \cos\alpha\frac{\partial}{\partial\beta}\right] + \left[\frac{\partial}{\partial\gamma}, \frac{\sin\alpha}{\sin\beta}\frac{\partial}{\partial\gamma}\right]\right) = 0$$

$$[J_3, L_z] = -\hbar^2\left(\left[\frac{\partial}{\partial\gamma}, \frac{\partial}{\partial\alpha}\right]\right) = 0$$

$$\therefore [J_3^2, L_{x,y,z}] = J_3[J_3, L_{x,y,z}] + [J_3, L_{x,y,z}]J_3 = 0$$

$\therefore$ 对称陀螺 $[\hat{H}, L_{x,y,z}] = 0$

④ Then prove general $[\hat{H}, L_{x,y,z}] = 0$

$$\hat{H} = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3} \qquad \because [J_3^2, L_{x,y,z}] = 0$$

$$\therefore [\hat{H}, L_{x,y,z}] = \left[\frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3}, L_{x,y,z}\right] = \left[\frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_1}, L_{x,y,z}\right]$$

令 $\hat{H}' = \dfrac{J_1^2}{2I_1} + \dfrac{J_2^2}{2I_2} + \dfrac{J_3^2}{2I_1}$

$\therefore [\hat{H}', L_{x,y,z}] = 0 \qquad \therefore [\hat{H}, L_x, L_y, L_z] = [\hat{H}', L_{x,y,z}] = 0$

⑤ Next prove $[J_1, J_2] = -i\hbar J_3, \quad [J_2, J_3] = -i\hbar J_1, \quad [J_3, J_1] = -i\hbar J_2$

1. $[J_1, J_2] = -\hbar^2\Big([\sin\gamma \frac{\partial}{\partial\beta}, \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}] + [\sin\gamma\frac{\partial}{\partial\beta}, -ctg\beta \sin\gamma \frac{\partial}{\partial\gamma}]$

$+ [-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}, \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\beta}] + [-\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}, \cancel{\frac{\sin\gamma}{\sin\beta}} - ctg\beta \sin\gamma\frac{\partial}{\partial\gamma}]$

$+ [ctg\beta\cos\gamma\frac{\partial}{\partial\gamma}, \cos\gamma\frac{\partial}{\partial\beta}] + [ctg\beta\cos\gamma\frac{\partial}{\partial\gamma}, \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}]$

$+ ctg^2\beta\, [\cos\gamma\frac{\partial}{\partial\gamma}, \sin\gamma\frac{\partial}{\partial\gamma}]\Big)$

$= -\hbar^2\frac{\partial}{\partial\gamma} = -i\hbar\hat{J}_3$

2. $[J_2, J_3] = -\hbar^2\Big([\sin\gamma\frac{\partial}{\partial\beta}, \frac{\partial}{\partial\gamma}] + [\frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}, \frac{\partial}{\partial\gamma}] + [-ctg\beta\sin\gamma\frac{\partial}{\partial\gamma}, \frac{\partial}{\partial\gamma}]\Big)$

$= -\hbar^2\Big(-\cos\gamma\frac{\partial}{\partial\beta} + \frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} - ctg\beta\cos\gamma\frac{\partial}{\partial\gamma}\Big) = -i\hbar\hat{J}_1$

3. $[J_3, J_1] = -\hbar^2\Big([\frac{\partial}{\partial\gamma}, \sin\gamma\frac{\partial}{\partial\beta}] + [\frac{\partial}{\partial\gamma}, -\frac{\cos\gamma}{\sin\beta}\frac{\partial}{\partial\alpha}] + [\frac{\partial}{\partial\gamma}, ctg\beta\cos\gamma\frac{\partial}{\partial\gamma}]\Big)$

$= -\hbar^2\Big(\big(-\cos\gamma\frac{\partial}{\partial\beta} - \frac{\sin\gamma}{\sin\beta}\frac{\partial}{\partial\alpha} + ctg\beta\sin\gamma\frac{\partial}{\partial\gamma}\Big) = -i\hbar\hat{J}_2$

⑥

$\langle 1\rangle: \hat{H} = \frac{J_1^2}{2I_1} + \frac{J_2^2}{2I_2} + \frac{J_3^2}{2I_3}$, $\qquad a \equiv \frac{1}{2I_1},\ b \equiv \frac{1}{2I_2},\ c \equiv \frac{1}{2I_3}$

$= \frac{1}{2}[(a+b)(J_1^2+J_2^2)] + cJ_3^2 + \frac{1}{2}(a-b)(J_1^2-J_2^2)$

$= \frac{1}{2}(a+b)(L^2 - J_3^2) + cJ_3^2 + \frac{1}{2}(a-b)(J_1^2 - J_2^2)$

Consider $(L^2, L_z, J_3)$ and its eigenstate $|l m k\rangle$. $\qquad (\because [L_z, J_3] = 0,\ [L^2, L_z] = 0$, $[L^2, J_3] = [L_x^2, J_3] + [L_y^2, J_3] + [L_z^2, J_3] = 0\ )$

$\hat{L}^2 |lmk\rangle = \hbar^2 l(l+1)|lmk\rangle$

$L_z|lmk\rangle = m\hbar|l, mk\rangle \qquad \hat{J}_3|l, m, k\rangle = k\hbar|lmk\rangle$

⟨2⟩ $J_+ \equiv J_1 + iJ_2$ , $J_- \equiv J_1 - iJ_2$

$\therefore (J_+^2 + J_-^2) = 2(J_1^2 - J_2^2)$

$H_1 = \frac{1}{2}(a+b)(L^2 - J_3^2) + CJ_3^2$ , $H_2 = \frac{1}{4}(a-b)(J_+^2 + J_-^2)$

$H \equiv \hat{H}_1 + \hat{H}_2$

$\hat{H}_{1,(l'm'k', lmk)} = \left\{\frac{1}{2}(a+b)[l(l+1)\hbar^2 - k^2\hbar^2] + ck^2\hbar^2\right\}$

$\delta_{ll'}\delta_{m'm}\delta_{kk'}$ $\therefore \hat{H}_1$ is diagonal, $\hat{H}_{1,lmk,lmk} = \langle lmk|\hat{H}_1|lmk\rangle$

$= \frac{\hbar^2}{2}(a+b)[l(l+1) - k^2] + ck^2\hbar^2$

⟨3⟩

$\because [J_3 J_+] = [J_3, J_1] + i[J_3, J_2] = -i\hbar J_2 + i(i\hbar)J_1 = i\hbar(J_1 + iJ_2)$

$= -\hbar J_+$

$[J_3 J_-] = [J_3 J_1] - i[J_3, J_2] = -i\hbar J_2 + \hbar J_1 = \hbar(J_1 - iJ_2) = \hbar J_-$

$= \hbar J_-$ $J_+^\dagger = J_-$ , $J_-^\dagger = J_+$ ;

$J_+^\dagger = J_1 - iJ_2 = J_-$

$\therefore [J_3, J_+]|imk\rangle = -\hbar J_+|imk\rangle$

$= J_3 J_+|imk\rangle - J_+J_3|i,mk\rangle$

$\therefore J_3 J_+|imk\rangle = (k-1)\hbar J_+|i,mk\rangle$ $\therefore J_+|imk\rangle$ has eigenvalue $(k-1)\hbar$

$J_+|imk\rangle \equiv c_1|i,m,k-1\rangle$

$\therefore [J_3, J_-]|imk\rangle = \hbar J_-|imk\rangle = J_3J_-|imk\rangle - J_-J_3|imk\rangle$

$\therefore J_3J_-|imk\rangle = (k+1)\hbar J_-|im,k\rangle$ $\therefore J_-|imk\rangle = c_2|i,m,k+1\rangle$

Compute $c_1, c_2$

$J_+J_- = (J_1 + iJ_2)(J_1 - iJ_2) = J_1^2 + J_2^2 + i(J_2J_1 - J_1J_2)$

$= L^2 - J_3^2 - \hbar J_3$

$J_-J_+ = (J_1 - iJ_2)(J_1 + iJ_2) = J_1^2 + J_2^2 + i(J_1J_2 - J_2J_1)$

$= L^2 - J_3^2 + \hbar J_3$

$|J_+|imk\rangle|^2 = \langle imk|J_-J_+|imk\rangle = (l(l+1) - k^2 + k)\hbar^2$

$= |C_1|^2$ $\therefore$ $C_1 = \sqrt{(l+k)(l-k+1)}\,\hbar$

$$\left|J_-|imk\rangle\right|^2 = \langle imk|J_+J_-|imk\rangle = (l(l+1)-k^2-k)\hbar^2 = |C_2|^2$$

$$C_2 = \sqrt{(l+k+1)(l-k)}\,\hbar$$

$$\therefore \quad J_+|imk\rangle = \sqrt{(l+k)(l-k+1)}\,\hbar\,|i,m,k-1\rangle$$

$$J_-|imk\rangle = \sqrt{(l-k)(l+k+1)}\,\hbar\,|i,m,k+1\rangle$$

$$\therefore \quad J_+^2|imk\rangle = \sqrt{(l+k)(l-k+1)}\,\hbar\,J_+|i,m,k-1\rangle$$

$$= \sqrt{(l+k)(l-k+1)(l+k-1)(l-k+2)}\,\hbar^2\,|i\,m,k-2\rangle$$

$$J_-^2|imk\rangle = \sqrt{(l-k)(l+k+1)(l-k-1)(l+k+2)}\,\hbar^2\,|i\,m,k+2\rangle$$

$\therefore H_2 = \frac{1}{4}(a-b)(J_+^2+J_-^2)$ is nonzero if $\Delta k = \pm 2$

$$\therefore \langle i' m' k'|H_2|i,m,k\rangle = \frac{1}{4}(a-b)\hbar^2\Big[\sqrt{(l+k)(l-k+1)(l+k-1)(l-k+2)}\,\delta_{k',k-2}$$

$$+\sqrt{(l-k)(l+k+1)(l-k-1)(l+k+2)}\,\delta_{k',k+2}\Big]\,\delta_{i'i}\,\delta_{m'm}$$

$$\therefore \quad \langle i'm'k'|H|imk\rangle = \langle i'm'k'|H_1|imk\rangle + \langle i'm'k'|H_2|imk\rangle$$

(4) We know $\hat{L}^2$'s eigvalue $l(l+1)\hbar^2$, $l = 0, 1, 2, 3 \cdots\cdots$

$\hat{L}_z$'s eigvalue $m\hbar$, $m = 0, \pm 1, \cdots \pm l$, ($l$已取定)

Now discuss $\hat{J}_3$'s quantum# $k$

$|k| \le \sqrt{l(l+1)}$, $\therefore k$ is bounded

Consider eigenstate $|l\,m\,k\rangle$, apply $J_+$ $\quad J_+|lmk\rangle, J_+^2|lmk\rangle, \cdots$

Corresponding eigenvalue $k\hbar$, $\cdots\cdots$ $(k-1)\hbar$ $(k-2)\hbar \cdots\cdots$

Eigenstate with lowest eigenvalue $|l,m,k_0\rangle$,

$J_+|imk_0\rangle = 0$

$\therefore |J_+|imk_0\rangle|^2 = (l(l+1)-k_0^2+k_0)\hbar^2 = 0 \Rightarrow (l+k_0)(l-k_0+1) = 0$

$\therefore k_0 = -l$ or $k_0 = l+1$ $\quad \because |k_0| \le \sqrt{l(l+1)}$ $\quad \therefore k_0 = -l$

$\because |J_+|i,mk\rangle|^2 \ge 0 \Rightarrow l+1 \ge k \ge -l$, $\therefore k$'s minimal value is $-l$

Apply $J_-$ to $|l, m, -l\rangle$, whose eigenvalue $-l\hbar, (-l+1)\hbar, \cdots$

$(\because k < \sqrt{l(l+1)})$

$k = 0, \pm 1, \pm 2, \cdots \pm l,$

<5> Now solve $H\psi = E\psi$

⓪ Fix $l$, the energy is $m$ independent, $\phi_{lmk} = |lmk\rangle \equiv \phi_k$

$$\psi = \sum_{k=-l}^{l} a_k \phi_k$$

$$\therefore \sum_{k=-l}^{l} a_k \hat{H} \phi_k = \sum_{k=-l}^{l} a_k \phi_k$$

$$\therefore \sum_{k=-l}^{l} a_k (\phi_{k'}, \hat{H}\phi_k) = a_{k'} E \quad (k' = -l, -l+1, \cdots l)$$

$\therefore l=0 \quad a_0(\phi_0 \hat{H} \phi_0) = a_0 E \quad \because a_0 \neq 0 \ \therefore E = H_{0m0,0m0} = 0$

$l=1 \quad \because l.m \quad l'.m'$ are fixed $H_{lmk,lmk} \longrightarrow H_{kk'}$

$H_{-1,-1} = (\frac{1}{2}(a+b)+c)\hbar^2, \ H_{00} = (a+b)\hbar^2, \ H_{11} = (\frac{1}{2}(a+b)+c)\hbar^2$

$H_{-1,1} = \frac{1}{2}(a-b)\hbar^2, \quad H_{1,-1} = \frac{1}{2}(a-b)\hbar,$

$$\therefore \det(H - EI) = 0 \Rightarrow \begin{vmatrix} H_{-1-1}-E & 0 & H_{-11} \\ 0 & H_{00}-E & 0 \\ H_{1-1} & 0 & H_{11}-E \end{vmatrix} = 0$$

$$(H_{-1-1}-E)(H_{00}-E)(H_{11}-E) - H_{-11}H_{1-1}(H_{00}-E) = 0$$

$$(H_{00}-E)((H_{11}-E)^2 - H_{1-1}^2) = 0 \quad (\because H_{-1-1} = H_{11}, \ H_{1-1} = H_{-11})$$

$$\therefore E_1 = H_{00} = (a+b)\hbar^2 = (\frac{1}{2I_1} + \frac{1}{2I_2})\hbar^2$$

$$E_2 = H_{11} + H_{-1,1} = (\frac{1}{2I_1} + \frac{1}{2I_3})\hbar^2 = (a+c)\hbar^2$$

$$E_3 = H_{11} - H_{-1,1} = (b+c)\hbar^2 = (\frac{1}{2I_2} + \frac{1}{2I_3})\hbar^2$$

① Proof $[J_i, J_j] = -i\epsilon_{ijk} J_k$

$$[L_a e_{i,a}, L_b e_{j,b}] = L_a [e_{i,a} L_b] e_{j,b} - L_b [e_{j,b}, L_a] e_{ia}$$

$$+ [L_a, L_b] e_{i,a} e_{jb}$$

$$= i L_a \epsilon_{abc} e_{i,c} e_{j,b} - i \underset{\downarrow a}{L_b} \epsilon_{bac} \underset{\downarrow b}{e_{jc}} \underset{\downarrow c}{e_{ia}} + i \epsilon_{abc} \underset{\downarrow a}{L_c} \underset{\downarrow c}{e_{ia}} e_{jb}$$

$$= i e_{ic} e_{jb} [\epsilon_{abc} L_a - L_a \epsilon_{acb} + L_a \epsilon_{cba}]$$

$$= i e_{ic} e_{jb} L_a \epsilon_{abc}$$

$$= -i \underbrace{\epsilon_{cba} e_{ic} e_{jb}} L_a$$

$$= -i (\vec{e}_i \times \vec{e}_j) \cdot \vec{L}$$

$$= -i \epsilon_{ijk} L_k$$

We used $[L_a \; e_{i,b}] = i \epsilon_{abc} e_{i,c}$

since body frame axis is a 3-vector under rotation