Lect 6 Lie's 1st, 2nd, 3rd therems, Lie algebra

§ Lie's 1rst therem — how infinitsmall element determines the Lie group?

Theorem one: the linear representatin of a conneted Lie group is deter[mined] by its generators.

Proof: consider $RS = T \longrightarrow t_j = f_j(r;s)$

then $D(R) = D(T)\,D(S^{-1})$, take derivatives with respect to r

$$\frac{\partial D(R)}{\partial r_k} = \frac{\partial D(T)}{\partial r_k} D(S^{-1}) = \sum_t \frac{\partial D(T)}{\partial t_j} \frac{\partial f_j(r;s)}{\partial r_k} D(S^{-1})$$

set $S = R^{-1}$, and take $T = E$ for $\frac{\partial D(T)}{\partial t_j}$, we have

$$\boxed{\begin{aligned} &\frac{\partial D(R)}{\partial r_k} = -i\left(\sum_j I_j\, S_{jk}(r)\right) D(R), \text{ with} \\ &I_j = i\,\frac{\partial D(T)}{\partial t_j}\bigg|_{T=E} \quad \text{and} \quad S_{jk}(r) \equiv \frac{\partial f_j(r;s)}{\partial r_k}\bigg|_{s=\bar{r}} \end{aligned}}$$

$S_{jk}(r)$ is determined matrix function, actually it's non-singular (it's the group element transformation, its determinaint is the Jocabian of valum element transformation. I_j's are the generaters. we have a first order differential matrix equation, with the initial condition $D(E) = I$. If we want to know the matrix at group element R, we can integrate the evolution equation follow a path from $E \to R$.

Question: if we have two paths, should we obtain consistent results?

Example: SO(2) group representation

: We take the rotation angle as the group parameter

$$f(\omega_1, \omega_2) = \omega_1 + \omega_2, \qquad S(\omega_1) = \left.\frac{\partial f(\omega_1, \omega_2)}{\partial \omega_1}\right|_{\omega_2 = \bar{\omega}_1} = 1$$

$$\begin{cases} \dfrac{\partial D(\hat{n}, \omega)}{\partial \omega} = -i I \, D(\hat{n}, \omega) \\ \text{with } D[\hat{n}, 0] = 1 \end{cases} \Rightarrow D[\hat{n}, \omega] = e^{-iI\omega}$$

If we diagonalize I, then $D[\hat{n}, \omega]$ is a direct sum of 1D representations

According to $D(\omega + 2\pi) = D(\omega)$ $\Rightarrow$ eigenvalues of I should be integer

$$\Rightarrow D^{(m)}(\hat{n}, \omega) = e^{-im\omega}.$$

If we know the rotation axis $\hat{n}$, $\rightarrow I = \sum I_a n_a$

$$\Rightarrow D[\hat{n}, \omega] = e^{-iI_a n_a \omega} = e^{-iI_a \omega_a}.$$

- Corollary
  - If for 2 representations, their generator $\bar{I}_j = X^{-1} I_j X$, then the 2 representations are equivalent
- An irreducible representation, $\Leftrightarrow$ if a matrix commutable with all the generators, it must be a constant matrix.
- $I_j^{(1)}$ and $I_j^{(2)}$ are two non-equivalent irreducible representation generators of their dimensions m_1, and m_2. If there exist $m_1 \times m_2$ matrix X such that $I_j^{(1)} X = X I_j^{(2)}$ then $X \equiv 0$.

Theorem 2: The generaters of the linear representations of Lie group satisfy the commutation relation

$$I_j I_k - I_k I_j = i \sum_l C^l_{jk} I_l \text{, where } C^l_{jk} = \left\{ \frac{\partial S_{lk}(r)}{\partial r_j} - \frac{\partial S_{lj}(r)}{\partial r_k} \right\}_{r}$$

Conversely, if there exist g matrices satisfy the above relation, then they can be used as a set of generaters of Lie group.

C^l_{jk} is a set of real numbers, which is independent of concrete Reps.

Prove: Start with $\dfrac{\partial D(R)}{\partial r_k} = -i \left(\sum_j I_j S_{jk}(r) \right) D(R)$ ← $\boxed{\begin{array}{l} D = TS^{-1} \\ t_j = f_j(r;s) \end{array}}$

where $I_j = i \dfrac{\partial D(T)}{\partial t_j}\Big|_{T=E}$, and $S_{jk}(r) = \dfrac{\partial f_j(r;s)}{\partial r_k}\Big|_{s=\bar{r}}$.

Then

$$\frac{\partial^2 D(R)}{\partial r_j \partial r_k} = -i \sum_l I_l \frac{\partial S_{lk}(r)}{\partial r_j} D(R) - i \sum_l I_l S_{lk}(r) \frac{\partial D(R)}{\partial r_j}$$

$$= -i \sum_l I_l \frac{\partial S_{lk}(r)}{\partial r_j} D(R) - \sum_l I_l S_{lk}(r) I_p S_{pj}(r) D(R)$$

exchange j and k

$$\frac{\partial^2 D(R)}{\partial r_k \partial r_j} = -i \sum_l I_l \frac{\partial S_{lj}(r)}{\partial r_k} D(R) - \sum_l I_l I_p S_{lj}(r) S_{pk}(r) D(R)$$

If the solution of $D(R)$ is unique, along any path (topo-equivalent) we should have

$$\frac{\partial^2 D(R)}{\partial r_j \partial r_k} = \frac{\partial^2 D(R)}{\partial r_k \partial r_j} \Rightarrow$$

$$\sum_{lp} [I_l I_p - I_p I_l] S_{lj}(r) S_{pk}(r) = i \sum_l I_l \left[\frac{\partial S_{lk}(r)}{\partial r_j} - \frac{\partial S_{lj}(r)}{\partial r_k} \right]$$

multi-ply the inverse $\bar{S}_{jl'}(r)\,\bar{S}_{kp'}(r)$ $\Rightarrow$

$$I_{l'}I_{p'} - I_{p'}I_{l'} = i\sum_{l} I_l \left[\frac{\partial S_{lk}(r)}{\partial r_j} - \frac{\partial S_{lj}(r)}{\partial r_k} \right] \bar{S}_{jl'}(r)\,\bar{S}_{kp'}(r)$$

or $I_j I_k - I_k I_j = i\sum_{l} I_l\, C^l_{jk}$, with $C^l_{jk} \doteq \sum_{pq} \left(\frac{\partial S_{lq}(r)}{\partial r_p} - \frac{\partial S_{lp}(r)}{\partial r_q} \right) \bar{S}_{pj}(r)\,\bar{S}_{qk}(r)$

Since LHS is independent of r, so does C^l_{jk}, set $r = E$, $\Rightarrow$

$$C^l_{jk} = \sum_{pq} \left\{ \frac{\partial S_{lk}(r)}{\partial r_j} - \frac{\partial S_{lj}}{\partial r_k} \right\} \Big|_{r=0}.$$

I will skip the proof of the converse, which is tedious. But use SO(3) as an illustration:

For SO(3) $[L_a, L_b] = i \sum_d \epsilon_{abd} L_d$, then $C^d_{ab} = \epsilon_{abd}$.

Let use find matrices of L_a satisfying this relation.

Introduce $L_\pm = L_1 \pm i L_2$, then $[L_3, L_\pm] = \pm L_\pm$, $[L_+, L_-] = 2L_3$

$$L^2 = L_3^2 + L_3 - L_- L_+ = L_3^2 - L_3 + L_+ L_-, \text{ and } [L^2, L_a] = 0.$$

Denote $|m\rangle$ as L_3's eigenstate: $\begin{cases} L_3 |m\rangle = m|m\rangle \\ L_3 L_\pm |m\rangle = (m\pm1) L_\pm |m\rangle \end{cases}$

Hence for finite dimensional Rep, there exist a highest weight state

satisfing $\begin{cases} L_+|l\rangle = 0 \\ L_3|l\rangle = l \end{cases}$ $\Rightarrow$ $L^2 = l(l+1)$.

Starting from $|l\rangle$, successively apply L_-, such that

$$L_-^n |l\rangle \neq 0, \quad \text{but} \quad L_-^{n+1}|l\rangle = 0.$$

The $L_3 L_-^n |l\rangle = (l-n) L_-^n |l\rangle$, and then $L^2 L_-^n |l\rangle = [(l-n)^2 - (l-n)] L_-^n |l\rangle = l(l+1) L_-^n |l\rangle$

$\Rightarrow (\ell-n)(\ell-n-1) = \ell(\ell+1) \Rightarrow n = 2\ell$

hence $\ell = n/2$, which can be integer, or, half an integer.

Denote $L_- |m\rangle = A_m |m-1\rangle$, $-\ell \le m \le \ell$

$$\langle m| L_+ L_- |m\rangle = \langle m| L^2 - L_3^2 + L_3 |m\rangle = |A_m|^2$$

$$A_m^2 = \ell(\ell+1) - m^2 + m = (\ell+m)(\ell-m+1),$$ use the convention $A_m > 0$

$\Rightarrow$

$$L_- |m\rangle = \sqrt{(\ell+m)(\ell-m+1)}\; |m-1\rangle$$

similarly $L_+ |m\rangle = \sqrt{(\ell-m)(\ell+m+1)}\; |m+1\rangle$

This set of L_3, $L_\pm$, are just what we derive from the Lie group D-matrix by taking derivatives. Hence Solving the Lie group problem is reduced to solve the Lie algebra.

Theorem 3: the structure constant

① $C_{jk}^{\ell} = -C_{kj}^{\ell}$ antisymmetric by switching j and k

② The generators should satisfy the Jocab identity

$$[[I_j, I_k] I_\ell] + [[I_k I_\ell] I_j] + [[I_\ell, I_j] I_k] = 0$$

$$\Rightarrow \sum_p \left\{ C_{jk}^{p} C_{p\ell}^{q} + C_{k\ell}^{p} C_{pj}^{q} + C_{\ell j}^{p} C_{pk}^{q} \right\} = 0$$

The structure constants of a Lie group satisfy the above relations.

Conversely, for a set of C_{jk}^{ℓ} satisfying these relations, we can also construct a Lie group.

Lie groups share the same structur constants have the same local structure, but can have different global structure.

For example:
$U(2)$ locally $\sim U(1) \times SU(2)$, but globally $U(2) \neq U(1) \otimes SU($

Since $\pm I \in$ both $U(1)$, and $SU(2)$.

$$U(2): \quad e^{i\varphi} \begin{bmatrix} h_0 - ih_3 & -h_2 - ih_1 \\ h_2 - ih_1 & h_0 + ih_3 \end{bmatrix}.$$

$\varphi \to \varphi + \pi$ can be the same as $h \to -h$.

hence $U(2) = U(1) \otimes SU(2) / Z_2$

(half quantum vortex — Alice string)

§ The adjoint Rep:

$$D(R)\, I_j\, D^{-1}(R) = \sum_k I_k\, D^{ad}_{kj}(R),$$

take R as infinitesmall elements.

$$D(R) = 1 - i \sum_l r_l I_l, \quad D^{-1}(R) = 1 + i \sum_l r_l I_l$$

$$\Rightarrow \quad D^{ad}_{kj}(R) = \delta_{kj} - i \sum_l r_l (I^{ad}_l)_{kj}$$

$$\Rightarrow \quad I_j - i \sum_l r_l I_l I_j + i \sum_l r_l I_j I_l = I_j - i \sum_l r_l I_k (I^{ad}_l)_{kj}$$

$$\Rightarrow \quad \cancel{-i}[I_l, I_j] = I_k (I^{ad}_l)_{kj} \quad \Rightarrow \quad (I^{ad}_l)_{kj} = iC^k_{lj}$$

or $\quad$ or $\boxed{(I^{ad}_j)_{kl} = iC^k_{jl}}$

HW: use Lie's 3rd theorem, prove that $(I^{ad}_j)_{kl} = iC^k_{jl}$ indeed satisfy the commutation relation $[I^{ad}_j, I^{ad}_k] = i C^p_{jk} I^{ad}_p$

For example, for SU(2) and SO(3) $C^{k}_{jl} = \epsilon_{kjl}$

$$\Rightarrow (I^{ad}_{a})_{bd} = i\, C^{b}_{ad} = i\, \epsilon_{adb} = -i\, \epsilon_{abd} = (T_a)_{bd}$$

$$C^{b}_{\underline{ad}} = \epsilon_{\underline{ad}b}$$

The structure constants are actually parameter dependent. For compact Lie-group, it's always possible to find real parameters such that the structure factors are fully antisymmetry for its 3 indices. Then it means that the adjenit Rep can be written as ~~puty~~ purely imaginary anti-symmetric matrix.

⊛ Definition of Lie algebra

$$[(-iI_j), (-iI_k)] = \sum_{l} C^{l}_{jk} (-iI_l)$$

$(-iI_j)$ can be viewed as a set of basis, to span a real linear space. In such a space, we use above commutation relation as the definition of product. Hence such a space is closed for the product, hence form an algebra. It's a real Lie algebra.

If we allow the superposition coefficients to be complex, then it's complex Lie ~~ale~~ algebra. Different real Lie algebras can share the same complex Lie algebra. For example, the Lorentz group SO(3,1) has a real non-compact Lie algebra. SO(4) has a compact real algebra. Nevertheless, they share the same complex Lie algebra.

If a Lie group G has an invariant sub Lie group H, their Lie algebras are denoted as L_G and L_H, whose dimensions are g and h respectively. Assume H has h parameters, I_μ with $1 \le \mu \le h \in L_H$.

and $A(\alpha)$ is infintismall element of H, then

$$D(A) = 1 - i \sum_{\mu=1}^{h} \alpha_\mu I_\mu, \quad \text{for a } R \in G,$$

$$D(R) = 1 - i \sum_{j=1}^{g} r_j I_j, \quad \text{then } D(R)\, D(A)\, D^{-1}(R) \in H$$

$$= 1 - i \sum_{j=1}^{g} \sum_{\mu=1}^{h} r_j \alpha_\mu [I_j, I_\mu]$$

$$= 1 - i \sum_{\nu=1}^{h} \beta_\nu I_\nu$$

Hence $[I_j, I_\mu] \in L_H$, for any $I_j \in L_G$ and $I_\mu \in L_H$.

Then L_H is called the ideal of L_G.

If a group does not have non-trival ideal, then it's call the simple Lie group! And its algebra is simple Lie algebra.

According to

$$\begin{cases} [I_l, I_j] = \sum_k I_k \, (I_l^{ad})_{kj} \\ D(R)\, I_j\, D^{-1}(R) = \sum_k I_k \, D^{ad}_{kj}(R) \end{cases}$$

The necessary & sufficient conditions for a Lie group to be a simple Lie group is that the adjont Rep is irreducible.

Casimir

If the structure factor is fully anti-symmetric, the sum of squre of each generators commute with each generater

$$[\sum_j I_j^2, I_k] = \sum_j I_j [I_j, I_k] + [I_j\, I_k] I_j$$

$$= i \sum_{jl} I_j I_l\, C_{jk}^{l} + I_l I_j\, C_{jk}^{l} = i \sum_{jl} I_j I_l [C_{jk}^{l} + C_{lk}^{j}] = 0$$

hence $\boxed{\sum_j I_j^2 = C_2(\lambda) \mathbb{1},}$ $C_2(\lambda)$ is a Rep dependent constant, which is call Casimir.

For compact simple Lie group, then its adjoint Rep is irreducible. We introduce $g \times g$ matrix $T_{jk} = Tr[I_j I_k]$ where I_j is the generater in the λ-representation. Then the adjoint Rep is also $g \times g$. It can proved that $[I_p^{ad}, T] = 0$, for any P.

HW: prove this!

Then T is a constant matrix $\Rightarrow$ $T_{jk} = Tr[I_j I_k] = \delta_{jk} T_2(\lambda)$ where $T_2(\lambda)$ is a constant. We only need to know the concrete form of one generater, then $T_2(\lambda)$ can be read out. Then from

$$Tr[I_j I_k] = \delta_{jk} T_2(\lambda) \Rightarrow \text{set } j=k, \text{ sum over } j$$

$$\Rightarrow Tr[\sum_j I_j^2] = T_2(\lambda)\, g = C_2(\lambda)\, m_\lambda \Rightarrow \boxed{C_2(\lambda) = T_2(\lambda) \cdot g / m_\lambda}$$

Example: For $SO(3)$, $[I_j, I_k] = i\,\epsilon_{jkl} I_l$

Then $\bar{T}_{jk} = tr[I_j I_k] = \delta_{jk} T_2(l)$

take the digonal $tr[I_3^2] = 2\sum_{m=1}^{l} m^2 = l(l+1)(2l+1)/3 = T_2(l)$

Then $g\, T_2(\lambda) = 3 \cdot \frac{l(l+1)(2l+1)}{3} = C_2(\lambda)\cdot(2l+1)$

$\Rightarrow C_2(\lambda) = l(l+1)$.

HW: work out the Casimir for half-integer spin Representation of $SU(2)$