Lect 3: Integral over group space (manifold)

We have learned group function $F(R)$ for $R \in G$, for finite groups. For example, $F(R)$ can be the character of R, etc. For the characters $\chi(R)$, we have

$$\frac{1}{g} \sum_{R} \chi_A^*(R)\, \chi_B(R) = \delta_{AB} .$$

Now we need to consider how to generalize this kind of results to continuous Lie groups. We can imagine

$$\frac{1}{g} \sum_{R \in G} F(R) \longrightarrow \int dR = \int dr\, W(R),$$

where "r" is parameter of the group element $R(r)$. $W(R)$ is weight function: In the neighborhood of R, is the volume of dr, the group element density. As a density function, we require

$$\boxed{\begin{aligned} &\int dR = \int dr\, W(R) = 1 \quad \leftarrow \text{normalization} \\ &\int dR\, F(R) = \int dr\, W(R)\, F(R) > 0, \quad \text{if } F(R) \geq 0 \text{ but not } \equiv 0. \end{aligned}}$$

The integral of the group function over the group manifold (group space) is call the integral over the group. It's a linear operation

$$\boxed{\int dR\, (a F_1(R) + b F_2(R)) = a \int dR\, F_1(R) + b \int dR\, F_2(R)}$$

The integral over the group manifold also satisfies

$$\boxed{\begin{array}{l}\int dR\ F(R) = \int dR\ F(SR) = \int dR\ F(RS) = \int dR\ F(R^{-1}) \\ \text{Ex: prove these relations — start from finite groups.}\end{array}}$$

Now we consider how to work out the weight function $W(R)$ under a certain set of parametrizations. Suppose near the identity, the group elements denoted as A with parameters α_j, the weight W_0 is set as a reference. Then in the neighborhood around the element R, the elements are denoted as R' with parameters denoted r_j'. Then

since $\underset{\text{variable}}{R'} = \underset{\text{variable}}{A}\overset{\text{fixed}}{R} \Rightarrow r_j' = f_j(\alpha; r)$ ← Composition function

hence $W(R)\, dr_1' \wedge \cdots dr_j' = W_0\, d\alpha_1 \wedge \cdots d\alpha_j$

$$\Rightarrow \boxed{W_0 = W(R) \left| \det \left\{ \frac{\partial f_j(\alpha; r)}{\partial \alpha_k} \right\} \right|_{\alpha=0}}$$

Similarly, according to $A = R' R^{-1}$, we have

$$\boxed{W(R) = W_0 \left| \det \left\{ \frac{\partial f_j(r', \bar{r})}{\partial r_k'} \right\} \right|_{r'=r}}$$

§ SU(2) group integral

$$(d\mathcal{U}) = W(\hat{n},\omega)\, d\omega_1 d\omega_2 d\omega_3 = W(\hat{n},\omega)\, \omega^2 d\omega \sin\theta d\theta d\varphi$$

spherical coordinate for the group manifold

① Due to the isotropy of the space, $W(\hat{n},\omega)$ should be independent of $\hat{n}$, but only a function of ω.

② Use $\mathcal{U}(\vec{e}_3,\omega)$ as R, and $\mathcal{U}(A)$ as A. We only keep the linear order of α_j

$$\mathcal{U}(A) = 1 - i(\sigma_1\alpha_1 + \sigma_2\alpha_2 + \sigma_3\alpha_3)/2$$

$$\mathcal{U}(\vec{e}_3,\omega) = \cos\omega/2 - i\sigma_3 \sin\omega/2$$

$$\mathcal{U}(A)\,\mathcal{U}(\vec{e}_3,\omega) = \cos\frac{\omega}{2} - \frac{\alpha_3}{2}\sin\omega/2 - i\frac{\sigma_1}{2}\left[\alpha_1\cos\frac{\omega}{2} + \alpha_2\sin\frac{\omega}{2}\right]$$

$$- i\sigma_2\left[\alpha_2\cos\omega/2 - \alpha_1\sin\omega/2\right] - i\frac{\sigma_3}{2}\left[\alpha_3\cos\frac{\omega}{2} + 2\sin\omega/2\right]$$

$$\mathcal{U}[\hat{n}',\omega'] = \cos\frac{\omega'}{2} - i(\vec{\sigma}\cdot\hat{n}')\sin\frac{\omega'}{2}$$

$$\Rightarrow \cos\frac{\omega'}{2} = \cos\frac{\omega}{2} - \frac{\alpha_3}{2}\sin\frac{\omega}{2} \simeq \cos\left[\frac{\omega+\alpha_3}{2}\right]$$

$$\sin\frac{\omega'}{2} = \sin\left(\frac{\omega}{2} + \frac{\alpha_3}{2}\right) = \sin\frac{\omega}{2} + \frac{\alpha_3}{2}\cos\frac{\omega}{2}$$

$$\Rightarrow \quad n_1'\sin\frac{\omega'}{2} = \frac{1}{2}\left[\alpha_1\cos\frac{\omega}{2} + \alpha_2\sin\frac{\omega}{2}\right]$$

Since α_1, α_2 are small quantity, hence n_1' is small, we can neglecte the difference between ω and ω' $\Rightarrow$

$$n_1' = \left[\sin\frac{\omega}{2}\right]^{-1}\frac{1}{2}\left[\alpha_1\cos\frac{\omega}{2} + \alpha_2\sin\frac{\omega}{2}\right]$$

$$\boxed{\omega' n_1' = \omega\left[\sin\frac{\omega}{2}\right]^{-1}\frac{1}{2}\left[\alpha_1\cos\frac{\omega}{2} + \alpha_2\sin\frac{\omega}{2}\right]}$$

similaly $\boxed{\omega' n_2' = \omega \left[\sin\frac{\omega}{2}\right]^{-1} \frac{1}{2}\left\{ \alpha_2 \cos\omega/2 - \alpha_1 \sin \omega/2 \right\}}$.

Also

$$n_3' \sin\frac{\omega'}{2} = \frac{1}{2}\left[\alpha_3 \cos\frac{\omega}{2} + 2\sin\frac{\omega}{2}\right]$$

since n_1', n_2' are small, $n_3' = \sqrt{1-n_1'^2-n_2'^2} \simeq 1$

$$\Rightarrow \sin\frac{\omega'}{2} = \sin\frac{\omega}{2} + \frac{\alpha_3}{2}\cos\frac{\omega}{2} \simeq \sin\frac{\omega+\alpha_3}{2} \Rightarrow$$

$$\boxed{\omega' n_3 \simeq \omega' = \omega + \alpha_3}$$

$$\Rightarrow \frac{W_0}{W(\omega)} = \left| \det\left[\frac{\partial(\omega' n_1 \quad \omega' n_2 \quad \omega' n_3)}{\partial(\alpha_1 \quad \alpha_2 \quad \alpha_3)} \right] \right|$$

$$= \begin{vmatrix} \frac{\omega}{2}\cot\frac{\omega}{2} & \omega/2 & 0 \\ -\omega/2 & \frac{\omega}{2}\cot\omega/2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \left(\frac{\omega}{2}\right)^2 \left[\cot^2\frac{\omega}{2} + 1\right] = \frac{\omega^2}{4\sin^2\frac{\omega}{2}}$$

Hence

$$W(\omega) = W_0 \; 4\sin^2\frac{\omega}{2} / \omega^2$$

normalization

$$1 = W_0 \int_0^{2\pi} 4\sin^2\frac{\omega}{2}/\omega^2 \cdot \omega^2 d\omega \int_0^{\pi} \sin\theta d\theta \int_{-\pi}^{\pi} d\varphi$$

$$= W_0 \cdot 4 \cdot \pi \cdot 4\pi \Rightarrow W_0 = \frac{1}{16\pi^2}$$

Hence, if use $\vec{\omega}$ as parameter

$$\boxed{\int (du) = \frac{1}{4\pi^2} \int_0^{2\pi} \sin^2\frac{\omega}{2} d\omega \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi = \frac{\sin^2\frac{\omega}{2}}{4\pi^2\omega^2} d\omega_1 d\omega_2 d\omega_3}$$

HW: prove that for SO(3) group

$$(dR) = \frac{1}{2\pi^2} \sin^2\frac{\omega}{2} \sin\theta \, d\omega \, d\theta \, d\varphi$$

If ~~for~~ integrals for class functions, we can first integrate out

$\int \sin\theta \, d\theta \, d\varphi = 4\pi$.

(*) Compact Lie group $\longleftrightarrow$ finite group.

$$\int dR \ F(R) \longleftrightarrow \frac{1}{g} \sum_{g \in G} F(R)$$

The following results for finite groups remain true for the compact Lie groups.

① linear Rep is equivalent to the unitary Rep. and two unitary Reps can be related by unitary transformations;

② real Rep is equivalent to the orthogonal Rep.

③ reducible Rep $\Rightarrow$ complete reducible. The necessary & sufficient conditions for an irreducible Rep: there are no non-~~or~~ constant matrices commute with all the Representation matrices

④ orthogonality

$$\int dR \; D^{i\,*}_{\mu\rho}(R) \; D^{j}_{\nu\lambda}(R) = \frac{1}{m_j}\delta_{ij}\,\delta_{\mu\nu}\,\delta_{\rho\lambda}$$

$$\int dR \; \chi^{i\,*}(R) \; \chi^{j}(R) = \delta_{ij}$$

Expansion of a representation in terms of irreducible ones

$$X^{-1} D(R) X = \oplus_j \, a_j \, D^j(R)$$

$$\rightarrow \chi(R) = \sum_j a_j \chi^j(R) \Rightarrow a_j = \int dR \; \chi^{j\,*}(R)\,\chi(R)$$

$$\boxed{\Rightarrow \quad a_j = \frac{1}{\pi}\int_0^{2\pi} d\omega \, \sin^2\frac{\omega}{2} \; \chi^{j\,*}(\omega)\,\chi(\omega)}$$

⑤ The sufficient & necessary condition for an irreducible rep.

$$\int dR \; |\chi(R)|^2 = 1.$$

HW: For SU(2) group, the non-equilent irreducible representation satisfy

$$\frac{1}{\pi}\int_0^{2\pi} d\omega \, \sin^2\frac{\omega}{2} \; \chi^{j\,*}(\omega)\,\chi^{j}(\omega) = \delta_{ij}$$

Check for all the integer and half-integer spin reprensentations they are all irreducible Reps.

Hint $\quad D^j_{mn}(\mathcal{U}(\hat{z},\omega)) = e^{-i\omega m}\,\delta_{m,n}\,, \quad m=-j,\cdots j.$

6 Self-conjugate representation, i.e. $D^*(R) = X^\dagger D(g) X$.

If there exists a set of basis in which $D(R)$ is real, then we call $D(R)$ as a real representation; if there does not exist such a set of basis, we call $D(R)$ pseudo-real.

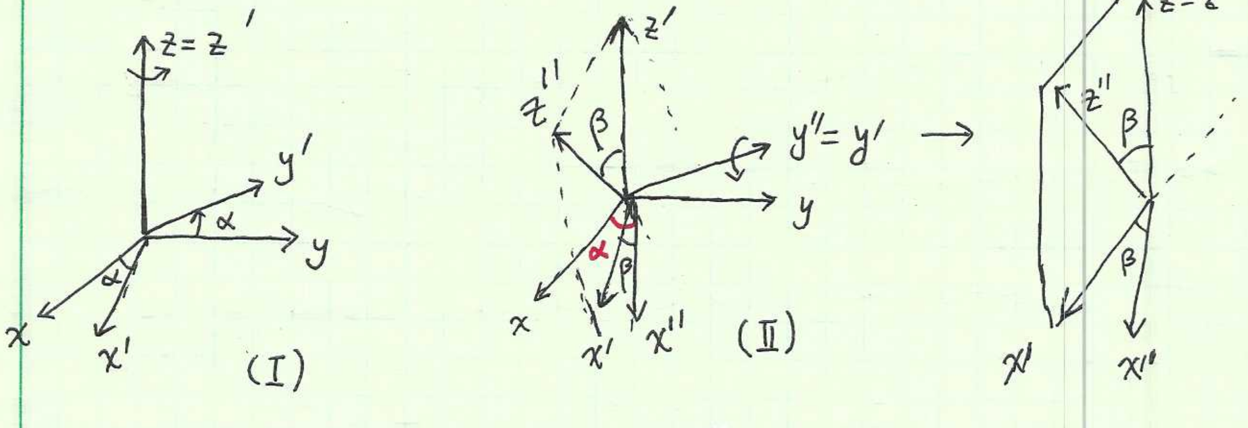
For real representations, X is symmetric matrix

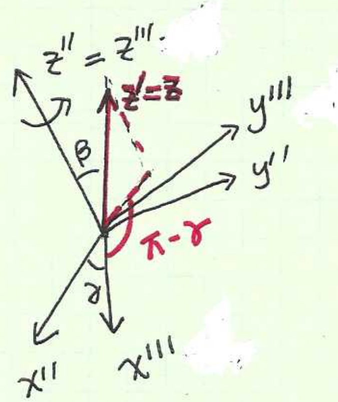
pseudo-real representation, X is anti-symmetric

And
$$\boxed{\int dR\ \chi(R^2) = \begin{cases} 0 & \text{if } D^* \not\sim D \\ +1 & \text{if } D^* \sim D \text{ and } D \text{ is real} \\ -1 & D^* \sim D \text{ and } D \text{ is pseudo-real} \end{cases}}$$

HW: For $SU(2)$ representations, Prove that those with integer spins are real, and those with half-integer spins are pseudo-real. Verify all the above results in point 6

(*) SU(2) group with Euler angles





(III)

x''', y''', z'' is the final configuration

$$R(\alpha, \beta, \gamma) = R(\hat{e}_3, \alpha)\, R(\hat{e}_2\, \beta)\, R(\hat{e}_3, \gamma)$$

$$= \begin{pmatrix} C_\alpha C_\beta C_\gamma - S_\alpha S_\gamma & -C_\alpha C_\beta S_\gamma - S_\alpha C_\gamma & C_\alpha S_\beta \\ S_\alpha C_\beta C_\gamma + C_\alpha S_\gamma & -S_\alpha C_\beta S_\gamma + C_\alpha C_\gamma & S_\alpha S_\beta \\ -S_\beta C_\gamma & S_\beta S_\gamma & C_\beta \end{pmatrix}$$

$C_\alpha = \cos\alpha$
$S_\alpha = \sin\beta$
and so on

$-\pi \le \alpha \le \pi$, $\quad 0 \le \beta \le \pi$, $\quad -\pi \le \gamma \le \pi$ for $SO(3)$

for $SU(2)$. $-2\pi \le \gamma \le 2\pi$

physical meaning of α, β, γ

$$R(\alpha\beta\gamma)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix} C_\alpha S_\beta \\ S_\alpha C_\beta \\ C_\beta \end{pmatrix},$$

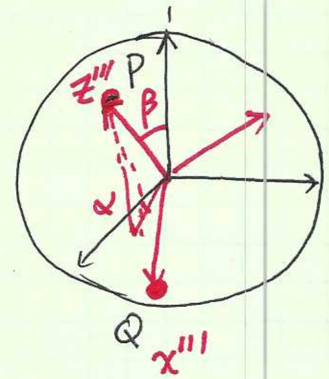
interprete as rotation in xyz frame

hence the $\hat{z}$-axis is rotated to z'''-axis ∘ the polar and azimuthal angle of the z'''-axis in the xyz-frame is just α and β.

we also have $R^{-1}(\alpha\beta\gamma)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\beta\cos(\pi-\gamma)\\ \sin\beta\sin(\pi-\gamma)\\ \cos\beta\end{pmatrix}$

we interpret this rotation in the $x'''y'''z'''$ frame. then $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is the z''' axis. R^{-1} rotates it back to z. Hence the azimuthal angle of $\hat{z}$ in the $x'''y'''z'''$ frame is $\pi-\gamma$.

⊛ measure



Consider a unit sphere: z''' intersects the unit sphere at P which determine α, β, and x''' intersects at Q.

If fix P, then Q's trajectory is a larg circle.

Around the group element $R_{(\alpha\beta\gamma)}$ the measure due to P is $\sin\beta d\beta d\alpha$ and the measure due to $Q \to d\gamma$, Hence

$$dR = c \sin\beta d\beta d\alpha d\gamma$$

$$1 = \int dR = c\int_0^{\beta}\sin\beta d\beta \int_{-\pi}^{\pi} d\alpha \int_{-\pi}^{\pi} d\gamma = 4\pi\cdot 2\pi \Rightarrow c = \frac{1}{8\pi^2} \text{ for } SO(3)$$

Similarly $\quad c = \frac{1}{16\pi^2}$ for $SU(2)$

Now we calculate the weight function $W(\alpha\beta\gamma)$, explicitly. The difficulty is that around $\beta \sim 0$, the correspondence from $\alpha\beta\gamma \to R(\alpha\beta\gamma)$ is many to one. Consider an element A near the identity with Eulerian angle parameters $(\alpha_0, \beta_0, \gamma_0)$. We take $\alpha_0 + \gamma_0 = r$, and $\beta_0 = t$ where r and t are infinitismal, but α_0 and γ_0 are finite. Then consider the group elements around the finite rotation $R(\alpha\beta\gamma)$, denoted as $R'(\alpha', \beta', \gamma')$. We set $A(\alpha_0\beta_0\gamma_0)\,R(\alpha\beta\gamma) = R'(\alpha'\beta'\gamma')$.

$$A(\alpha_0\beta_0\gamma_0) = \begin{pmatrix} 1 & -r & t\cos\alpha_0 \\ r & 1 & t\sin\alpha_0 \\ -t\cos\gamma_0 & t\sin\gamma_0 & 1 \end{pmatrix}$$

$$R(\alpha\beta\gamma) = \begin{pmatrix} C_\alpha C_\beta C_\gamma - S_\alpha S_\gamma & -C_\alpha C_\beta S_\gamma - S_\alpha C_\gamma & C_\alpha S_\beta \\ S_\alpha C_\beta C_\gamma + C_\alpha S_\gamma & -S_\alpha C_\beta S_\gamma + C_\alpha C_\gamma & S_\alpha S_\beta \\ -S_\beta C_\gamma & S_\beta S_\gamma & C_\beta \end{pmatrix}$$

① solve β' :
$$\cos\beta' = -t\cos\gamma_0\cos\alpha\sin\beta + t\sin\gamma_0\sin\alpha\sin\beta + \cos\beta$$
$$= \cos\beta - t\sin\beta\cos(\gamma_0+\alpha)$$
$$\Rightarrow \boxed{\beta' = \beta + t\cos(\gamma_0+\alpha)}$$

and $\sin\beta' = \sin\beta + \cos\beta\, t\cos(\gamma_0+\alpha)$

② solv $\alpha \Rightarrow \cos\alpha'\sin\beta' = \cos\alpha\sin\beta - r\sin\alpha\sin\beta + t\cos\alpha_0\cos\beta$

$$\cos\alpha' = \frac{\cos\alpha - r\sin\alpha + t\cos\alpha_0\cot\beta}{1 + \cot\beta\, t\cos(\gamma_0+\alpha)} \approx \cos\alpha - r\sin\alpha + t\cos\alpha_0\cot\beta - t\cot\beta\cos(\gamma_0+\alpha)\cos\alpha$$

$\alpha' = \alpha + \Delta\alpha \Rightarrow \cos\alpha' = \cos\alpha - \sin\alpha\,\Delta\alpha$

$$\Rightarrow \Delta\alpha = r - t\cot\beta \frac{1}{\sin\alpha}\Big[\overset{\cos\alpha_0}{} - \cos(\gamma_0+\alpha)\cos\alpha\Big]$$

replace $\cos\alpha_0$ by $\cos\gamma_0$, $\overset{\text{since}}{,}$ the coefficient t is already small, the error is neglected, then $\cos\gamma_0 - \cos(\gamma_0+\alpha)\cos\alpha = \cos\gamma_0[1-\cos^2\alpha] + \sin\gamma_0 \sin^2\alpha$

$$= \sin\alpha\,[\quad \sin(\alpha+\gamma_0)]$$

$$\Rightarrow \boxed{\alpha' - \alpha = r - t\cot\beta\sin(\alpha+\gamma_0)}$$

③ Solve γ'

$$-\sin\beta'\cos\gamma' = -t\cos\gamma_0[\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma] + t\sin\gamma_0[\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma) - \sin\beta\cos\gamma$$

$$= -t\cos\beta\cos\gamma\cos(\alpha+\gamma_0) - \sin\beta\cos\gamma - t\sin\gamma\sin(\alpha+\gamma_0)$$

$$\cos\gamma' = \frac{\cos\gamma + t[\cot\beta\overset{\cos(\alpha+\gamma_0)}{\cos\gamma} + \sin\gamma\sin(\alpha+\gamma_0)/\sin\beta]}{1 + t\cot\beta\cos(\gamma_0+\alpha)}$$

$$= \cos\gamma \quad + t\frac{\sin\gamma}{\sin\beta}\sin(\alpha+\gamma_0)$$

$$\Rightarrow \boxed{\gamma' = \gamma + \frac{t}{\sin\beta}\sin(\alpha+\gamma_0)}$$

plug in $r = \alpha_0 + \gamma_0$

$t = \beta_0$

$$\frac{\omega(\alpha_0\beta_0\gamma_0)}{\omega(\alpha\beta\gamma)} = \left|\det\frac{\partial(\alpha'\beta'\gamma')}{\partial(\alpha_0\beta_0\gamma_0)}\right| = \begin{vmatrix} 1 & -\cot\beta\sin(\alpha+\gamma_0) & 1 \\ 0 & \cos(\gamma_0+\alpha) & -t\sin(\gamma_0+\alpha) \\ 0 & \frac{\sin(\gamma_0+\alpha)}{\sin\beta} & \frac{t\cos(\gamma_0+\alpha)}{\sin\beta} \end{vmatrix}$$

$$= \frac{t}{\sin\beta} \simeq \frac{\sin\beta_0}{\sin\beta} \Rightarrow \boxed{(du) = \#\sin\beta\, d\beta\, d\alpha\, d\gamma}$$