Lect 3: Representation,

Consider a finite vector space V spanned by an orthonormal set of N vectors, i.e $\langle i|j\rangle = \delta_{ij}$, $\sum_{i=1}^{N} |i\rangle\langle i| = 1$. We represent the operation $g \in G$ by an $N \times N$ matrix

$$|i\rangle \longrightarrow |i(g)\rangle = \sum_j |j\rangle M_{ji}(g)$$

Then for a vector $|\psi\rangle = \sum_i a_i |i\rangle \Rightarrow g|\psi\rangle = \sum_i a_i |i(g)\rangle$

i.e. $$g|\psi\rangle = \sum_{ij} a_i |j\rangle M_{ji}(g) = \sum_j |j\rangle M_{ji}(a)\, a_i$$

express $g|\psi\rangle = \sum_j |j\rangle b_j \Rightarrow \boxed{b_j = M_{ji}(g)\, a_i}$

The operation g is mapped to a matrix $M(g)$. This maping preserves the multiplication table of the group G. Consider $g = g_1 g_2$, we have

group operation $g = g_1 \circ g_2$ $\xrightarrow{\text{homomorphism}}$ $M(g) = M(g_1) M(g_2)$ matrixes

$$|i(g)\rangle = \sum_j |j\rangle M_{ji}(g)$$

or $$|i\rangle \xrightarrow[g_2]{} |i(g_2)\rangle = \sum_j |j\rangle M_{ji}(g_2)$$

$$|i(g_1 g_2)\rangle = \sum_j |j(g_1)\rangle M_{ji}(g_2) = \sum_k |k\rangle M_{kj}(g_1) M_{ji}(g_2)$$

$$\Rightarrow M_{ki}(g) = \sum_j M_{kj}(g_1) M_{ji}(g_2) \Rightarrow M(g) = M(g_1) M(g_2).$$

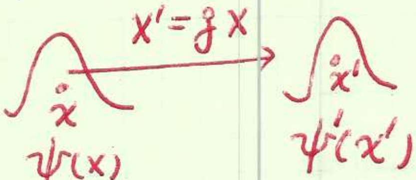
$$\boxed{\text{Excersice: Prove that } M(g^{-1}) = M^{-1}(g).}$$

(*) Quantum mechanical wavefunction for group representation

We view group elements as symmetry operations. When it applies to coordinates: $x \to x' = gx$, the scalar wavefunction $\psi \to \psi'$.

They satisfy the relation $\psi'(x') = \psi(x)$.

The operation on QM, wavefunction is represented by an operator P_R, i.e



$$\psi' \equiv P_g \psi, \quad \boxed{\begin{aligned} &\Rightarrow (P_g\psi)(x') = \psi(x) \\ &\text{or } (P_g\psi)(x) = \psi(g^{-1}x) \end{aligned}} \leftarrow \text{for } x' = gx$$

Example: ① translation $T(\delta) = e^{-i\hat{P}\delta} = e^{-\hbar\delta\frac{\partial}{\partial x}}$, $x \to x' = x + \delta$

For plane wave state $\psi_k(x) = e^{ikx}$: $\underbrace{-i\hbar\partial_k}_{\text{momentum operator}}(e^{ikx}) = \hbar k\, e^{ikx}$

$P = i\frac{\partial T(\delta)}{\partial \delta}\Big|_{\delta=0}$ ← infinitesimal generator

$$\underbrace{\left(T(\delta)\,\psi_k\right)(x)}_{e^{-\hbar\delta\frac{\partial}{\partial x}}\psi_k(x)} = \underbrace{\psi_k(x-\delta)}_{x' = x-\delta} = \boxed{e^{-ik\delta}}\, e^{ikx}$$

↖ character

- We can check that it satisfies the group product, $\boxed{P_{g_2}P_{g_1} = P_{g_2\cdot g_1}}$

Consider $g = g_2 \cdot g_1$, then $x'' = g_2 \cdot x' = g_2 \cdot g_1 x$

then $\psi(x) \xrightarrow{g_1} P_{g_1}\psi(x) = \psi(g_1^{-1}x)$

$\xrightarrow{g_2} P_{g_2}P_{g_1}\psi(x) = P_{g_1}\psi(g_2^{-1}x) = \psi(g_2^{-1}g_1^{-1}x) = \psi((g_1g_2)^{-1}x)$

$= P_{g_1g_2}\psi(x)$

★ How about linear operater? ~~ter~~

Consider an linear operater $O(x)$, It represents an operation

$$\psi_B(x) = O(x)\,\psi_A(x).$$

After coordinate transformation, $x' = gx$, the $\psi_{A,B}$ change

$$\psi_A(x) \xrightarrow{g} \psi_A'(x') = (P_g\psi_A)(x')$$

$$\psi_B(x) \xrightarrow{g} \psi_B'(x') = (P_g\psi_B)(x')$$

we define $\boxed{O(x) \xrightarrow{g} O'(x'), \text{ such that } \psi_B'(x') = O'(x')\,\psi_A'(x')}$

where $O'(x')$ is the transformed operater. The relation between O' and O $\Rightarrow$ $(P_g\psi_B)(x') = O'(x')\,(P_g\psi_A)(x')$

chang $x' \to x$: $(P_g\psi_B)(x) = O'(x)\,P_g\,\psi_A(x)$

$$\psi_B(x) = \underbrace{P_g^{-1}\,O'(x)}\,P_g\,\psi_A(x) = \underset{\sim}{O(x)}\,\psi_A(x)$$

$$\Rightarrow\quad O(x) = P_g^{-1}\,O'(x)\,P_g \quad\Rightarrow\quad \boxed{O'(x) = P_g\,O(x)\,P_g^{-1}}$$

or, more compact,

$$\langle\psi_A|\,O\,|\psi_B\rangle = \langle P_g\psi_A|\,O'\,|P_g\psi_B\rangle = \langle\psi_A|\,P_g^{\dagger}\,O'\,P_g\,|\psi_B\rangle$$

$$\Rightarrow\quad O' = P_g\,O\,P_g^{\dagger}$$

Consider a system with Hamiltonian $H(x)$, under the transformation

$$H(x) \xrightarrow{g} P_g H(x) P_g^{-1},$$

If g is a symmetry, i.e. $H(x)$ is invariant, $\Rightarrow$ $P_g H(x) P_g^{-1} = H(x)$

or $\boxed{[H(x), P_g] = 0.}$

$\Rightarrow$ Operator of a symmetry operation commutes with the Hamiltonian

If an energy level has $\underbrace{\text{degenercy}}_{m\text{-fold}}$, i.e.

$$H(x)\psi_\mu(x) = E\psi_\mu(x), \quad \mu = 1, 2, \cdots m.$$

Then $\{\psi_\mu(x)\}$ span a m-dimensional complex linear space, and $\psi_\mu(x)$'s form a set of basis.

$$H(x)\,(P_g\psi_\mu)(x) = P_g H(x)\psi_\mu = E\left(P_g\psi_\mu(x)\right)$$

$\Rightarrow$ $P_g\psi_\mu$ remains in the same space. i.e. the eigenfunctions with same energy form an invariant space of symmetry group operations.

$$\boxed{P_g\psi_\mu(x) = \sum_{\nu=1}^{m} \psi_\nu(x)\, D_{\nu\mu}(g).}$$

Then $D_{\nu\mu}(g)$ is the representation matrix. It can also be shown that $D(g_1 g_2) = D(g_1)D(g_2)$ and $D(g^{-1}) = D^{-1}(g)$. Hence we build up homomorphism $\boxed{g \approx P_g \sim D(g)}$

For example: rotation operation $P_g \sim R_{\hat{n}}(\theta) = e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}$.

$$|\psi_{\ell m}\rangle = R(r)\, Y_{\ell m}(\theta, \varphi)$$

$$e^{-i\vec{J}\cdot\hat{n}\theta/\hbar}|\psi_{\ell m}\rangle = \sum_{m'} |\psi_{\ell m'}\rangle \langle\psi_{\ell m'}| e^{-i\vec{J}\cdot\hat{n}\theta/\hbar} |\psi_{\ell m}\rangle$$

Define $D^{\ell}_{m'm'}(g) = \langle\psi_{\ell m'}| e^{-i\vec{J}\cdot\hat{n}\theta/\hbar} |\psi_{\ell m}\rangle$

$= \langle \ell m'| e^{-i\vec{J}\cdot\hat{n}\theta/\hbar} |\ell m\rangle \leftarrow$ D-matrix

Wigner function.

representation of SU(2), or SO(3) group.