Lect 10: More on SU(N) — Young tableau etc

⊛ decomposition of direct product

Litblewevel-Richarson's rule :

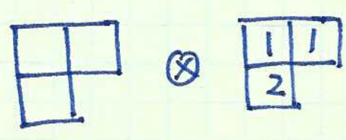
Consider a representation $[\lambda] \otimes [\mu]$. Let us take $[\lambda]$ as the reference, and often $[\mu]$ has less boxes. We fill $[\mu]$ with numbers, : ~~ja~~ number "j" for all the boxes in the j-th row. Then starting from the 1st row, we move the boxes in $[\mu]$ from up to the bottow to $[\lambda]$. It needs to satisfy the following requirements

① After finishing each row, we still have a regular Young pattern

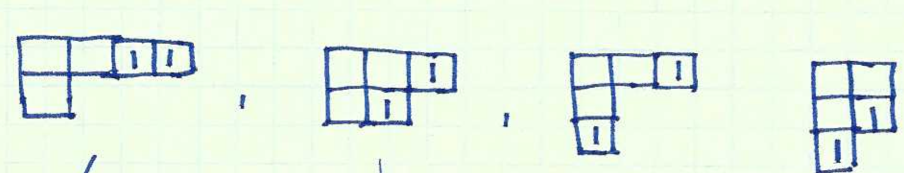
② the boxes with the same # are not in the same column

③ From the 1st row, row by row, from right → left, read out the new boxes, we require the # of boxes filled with large number ≤ # of boxes filled with small numbers during each step..

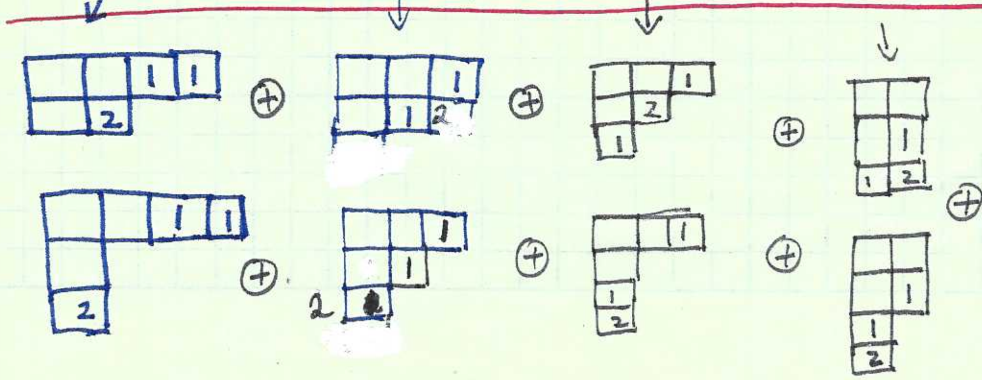
Example. $[21] \otimes [21]$.

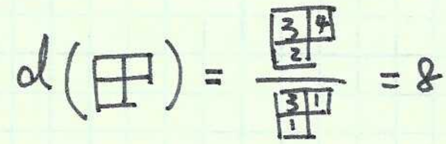


①

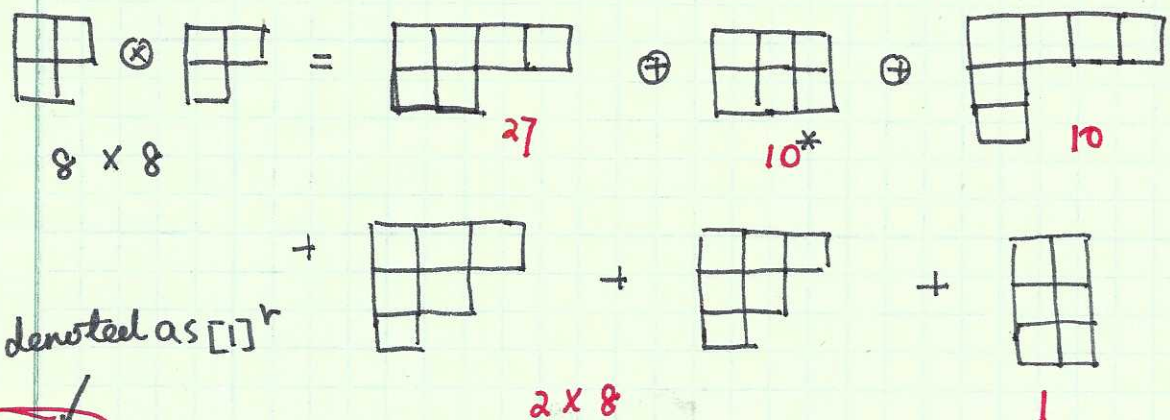


②

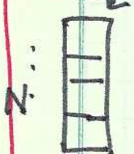


Apply the above for SU(3) group 

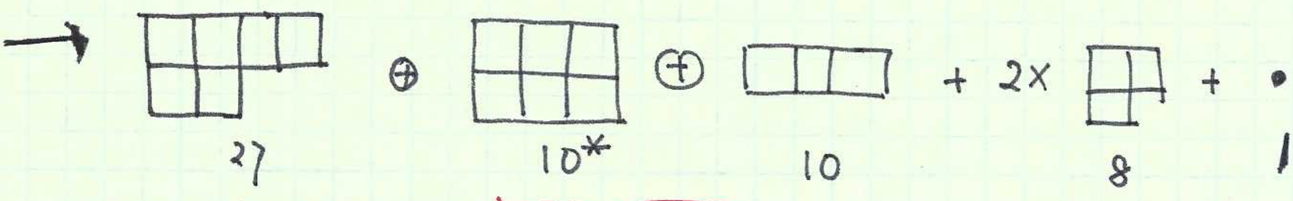
we should eleiminate rows beyond the 3rd lines, which corresponds to 0 dimensional space. Then we have



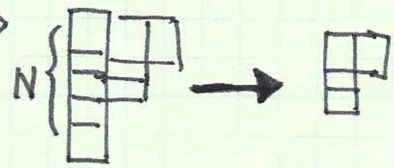
denoted as $[1]^r$



is an SU(N) singlet, which can be eliminated. HW: prove it



Complex conjugate representations

Hint: consider the direct product of $[1]^r \otimes [\lambda] = [\lambda] \Rightarrow$ 

④ covariant / contravariant tensors

If $D(G)$ is a representation of group G, then $D(G)^*$ or $[D^{-1}(G)]^T$ is also a representation. Tensors transform according to $(D(G))^*$ are called contravariant tensors.

$$O_u T^{a_1 \cdots a_n} = \sum_{b_1 \cdots b_n} u^*_{a_1 b_1} \cdots u^*_{a_n b_n} T^{b_1 \cdots b_n}$$

$$= \sum_{b_1 \cdots b_n} T^{b_1 \cdots b_n} (u^{-1})_{b_1 a_1} (u^{-1})_{b_2 a_2} \cdots (u^{-1})_{b_n a_n}$$

→ (n, m) mixed tensor

$$O_u T^{b_1 \cdots b_m}_{a_1 \cdots a_n} = \sum_{a', b'} u_{a_1 a_1'} \cdots u_{a_n a_n'} T^{b_1' \cdots b_m'}_{a_1' \cdots a_n'} (u^{-1})_{b_1' b_1} \cdots (u^{-1})_{b_m' b_m}$$

Contract: $(n, m) \to (n-1, m-1)$ ← trace tensor

$$O_u \sum_{c=1}^{N} T^{c\, b_1 \cdots}_{c\, a_1 \cdots} = \sum_{c d d'} u_{cd} u^*_{cd'} \sum_{a' b'} u_{a_1 a_1'} \cdots u^*_{b_1 b_1'} T^{d' b_1' \cdots}_{d\, a_1' \cdots}$$

$$= \sum_{(a')(b')} u_{a_1 a_1'} \cdots u^*_{b_1 b_1'} \cdots \left(\sum_{d} T^{d\, b_1' \cdots}_{d\, a_1' \cdots} \right)$$

* (1,1) mixed tensor − δ-function

$$O_u \delta^b_a = \sum_{a' b'} u_{aa'} u^*_{bb'} \delta^{b'}_{a'} = \sum_{a' b'} u_{aa'} \delta^{b'}_{a'} (u^{-1})_{b' b} = \delta^b_a$$

$\delta_a{}^b$ — decompose trace-space and traceless-space

$$T^b_a = \underbrace{\left\{ T^b_a - \tfrac{1}{N} \delta^b_a \sum T_c{}^c \right\}}_{\text{traceless space}} + \underbrace{\delta^b_a \left(\tfrac{1}{N} \sum T_c{}^c \right)}_{\text{trace}}$$

But for T^{d}_{ab} (2,1) rank-mixed tensor, it's more complicated

define
$$\Phi^{d}_{ab} = T^{d}_{ab} + \delta^{d}_{a}\left(\sum_{p=1}^{N} C_1 T_{b\,p}^{p} + C_2 T^{p}_{pb}\right) + \delta^{d}_{b}\left(\sum_{p=1}^{N} C_3 T_{a\,p}^{p} + C_4 T^{p}_{pa}\right)$$

according to $\sum_a \Phi^{a}_{ab} = 0$ and $\sum_b \Phi^{b}_{ab} = 0$, solve

$$C_1 = C_4 = 1/N^2-1, \quad C_2 = C_3 = -\frac{N}{N^2-1}.$$

④ Relation between covariant and contra-variant tensor

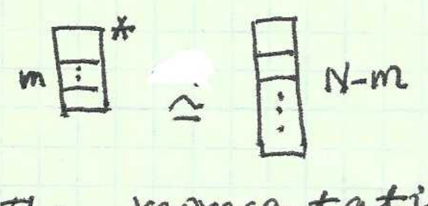
Consider m-th rank fully antisymmetric tensor $T^{b_1 \cdots b_m}$, i.e $[1^m]^*$,

we define
$$\Phi_{a_1 \cdots a_{N-m}} = \frac{1}{m!} \sum_{b_1 \cdots b_m} \epsilon_{a_1 \cdots a_{N-m} b_1 \cdots b_m} T^{b_1 \cdots b_m}$$

HW: Prove that $\Phi_{a_1 \cdots a_{N-m}}$ is a N-m rank fully anti-symmetric tensor, denoted as $[1^{N-m}]$. Also please find the inverse of the above expression, i.e use $\Phi_{a_1 \cdots a_{N-m}}$ to express $T^{b_1 \cdots b_m}$.

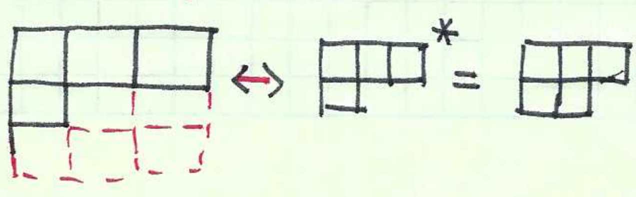
Hence Φ and T are two sets of bases of the same tensor subspace, hence $[1^m]^* \simeq [1^{N-m}]$.

i.e



We can similarly generalize to other representations.

For example



- Decomposition of SU(N) representation to SU(N−1)

Consider the problem to decompose rank-r tensor space of SU(N), i.e N-dim complex vector space into irreducible tensors. Each SU(N) rep is denoted by a Young pattern, and an irreducible tensor subspace is by a Young operator to a Young tableau filled with numbers of 1 to r. But when we use the Young tableau to construct the tensor basis, we are filling in each box with the index of basis vector from 1 to N. If we remove N, we get a basis for the SU(N−1) group.

For example: For the Rep $Y^{[3]}$ (fully symmetric), for SU(3), it's 10 dim.

they are

$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}$ → $\square\square\square$ 4

$\begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 2&2&3\\\hline\end{array}$ $\xrightarrow{\text{remov 3}}$ $\square\square$ 3

$\begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array}$ $\begin{array}{|c|c|c|}\hline 2&3&3\\\hline\end{array}$ → $\square$ 2

$\begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array}$ → $\bullet$ 1

$\underbrace{\qquad}_{SU(2)}$ — total: 10

another example: for Rep $Y^{[21]}$ of SU(3), it's 8 dimensional.

The basis

$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array}$ → $\begin{array}{|c|c|}\hline \ &\ \\\hline \ \\\cline{1-1}\end{array} = \square$ 2

remov $\boxed{3}$ →

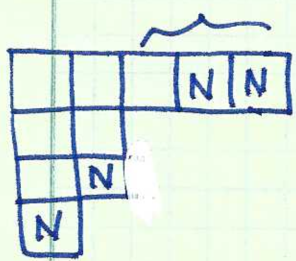
$\begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$ → $\square\square$, $\begin{array}{|c|}\hline \ \\\hline \ \\\hline\end{array} = \bullet$ +3 +1

$\begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$ → $\square$ +2

total: 8

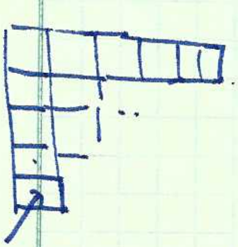
Basically, for a given Young pattern, (r-boxes), we fill in the index of vector basis to construct tensor subspace basis.

Each row can only be filled in at most one N. If an N indeed appears, we can perform vertical permutation, to move it to the end of the column. This permutation operation multiplies from the right to the Young operator, i.e, to the Q part, hence, it does not change the basis, just up to a sign.

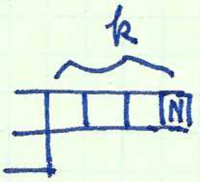


The for the first row: The only possible places for N's are those boxes without the 2nd row below it. Then we perform row permutation to move them to the upmost right boxes. Since they don't belong any columns, this row permutation commute with Q part of the Young operator, and does not change the basis.

Now we move N's to the right places: The ends of columns, and the first row from the right.



Two possiblies: with a N or not



: $k+1$ possibilities

No N,

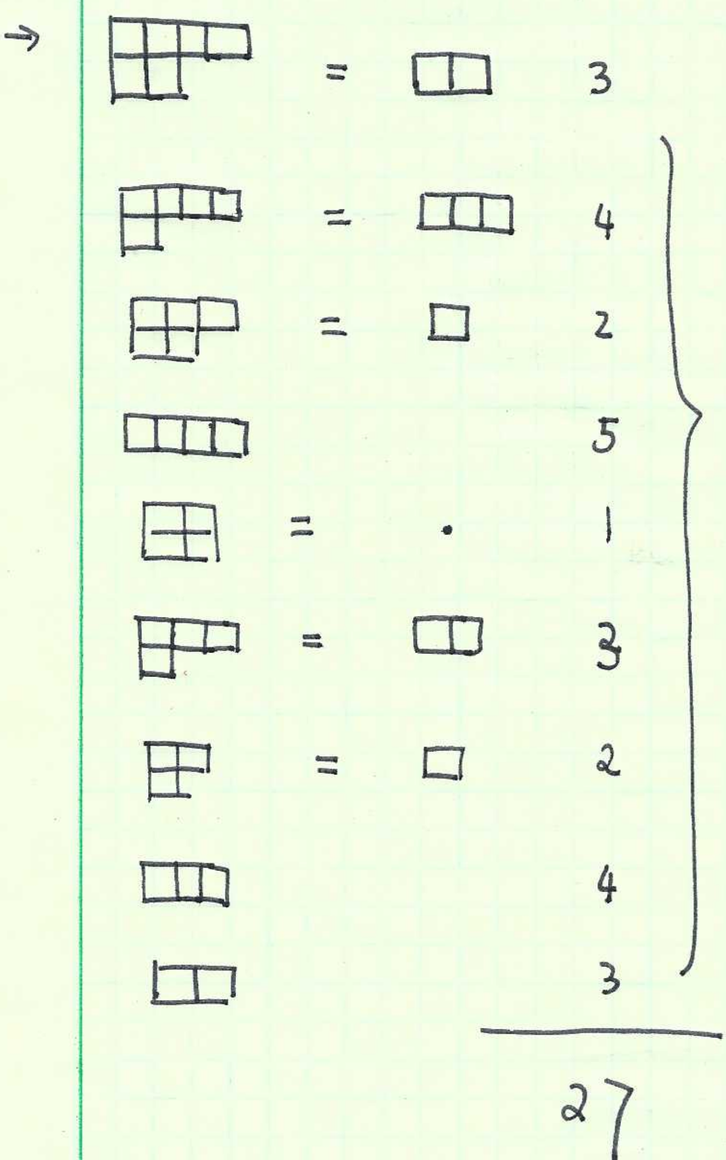
one N.

two N's

. . . k N's from right

Example: decompose the $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}$ representation of SU(3) in terms of SU(2).

Solution: The dimesion of $d\left(\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ \\ \cline{1-2}\end{array}\right) = \dfrac{\begin{array}{|c|c|c|c|}\hline 3 & 4 & 5 & 6 \\ \hline 2 & 3 \\ \cline{1-2}\end{array}}{\begin{array}{|c|c|c|c|}\hline 5 & 4 & 2 & 1 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}} = \dfrac{3\cancel{4}\cancel{5}\overset{3}{\cancel{6}}\cancel{2}3}{\cancel{5}\cancel{4}\cancel{2}\cdot\cancel{2}} = 27$



SU(2)

great!

This is a complete decomposition!

Decomposition of $\overbrace{\text{a Rep of}}$ $SU(M+N)$ into $SU(N) \otimes SU(M)$.

For example, in the grand unification theory $SU(5)$, its fundematal spinor $\square$ can be decomposed into a $[\underset{SU(3)}{\square} \otimes \underset{SU(2)}{\bullet}] \oplus [\underset{SU(3)}{\bullet} \otimes \underset{SU(2)}{\square}] = (3,1) \oplus (1,2)$

Suppose we have an $SU(N+M)$ representation, say. $SU(2+3)$'s representation of $\underset{r=5}{\ydiagram{2,1,1,1}}$. Let's decomposite into $SU(N) \otimes SU(M)$.

$N+M=5$ with $N=2$ and $M=3$.

we assume there are r_1 boxes for the $SU(N)$ and r_2 boxes for $SU(M)$ with $r_1+r_2=5$. Say a $\underbrace{\ }_{r_1=3}$ boxes $\ydiagram{1,1,1}$ for $SU(3)$, and $r_2=2$ $\ydiagram{1,1}$ for $SU(2)$. The number of the representation $\underset{SU(3)}{\ydiagram{1,1,1}} \otimes \underset{SU(2)}{\ydiagram{1,1}}$ in the representation $\underset{SU(3+2)}{\ydiagram{2,1,1,1}}$ is the number defined in the following way

vector bases 1 2 3

vector bases 4.5

we ~~treat~~ $\ydiagram{1,1,1}$ and $\ydiagram{1,1}$ as $SU(5)$ Reps, and we check the # of $\ydiagram{2,1,1,1}$ appears in $\ydiagram{1,1,1} \otimes \ydiagram{1,1}$. This # is the number of $\underset{SU(3)}{\ydiagram{1,1,1}} \otimes \underset{SU(2)}{\ydiagram{1,1}}$ appearing in the decompositing of $\underset{SU(5)}{\ydiagram{2,1,1,1}}$

How, we have