Lect 8 Applications of SU(3)

§1 Bilinear - biquadratic spin - chain

We know spin - 1 of SU(2) is a triplet Rep., and the fundament[al] Rep of SU(3) is also a triplet. What's the relation between them? Consid[er] a spin chain problem (spin - 1).

$$\boxed{H = J \sum_i \left\{ \cos\theta \, \vec{S}_i \cdot \vec{S}_{i+1} + \sin\theta \, (\vec{S}_i \cdot \vec{S}_{i+1})^2 \right\}}$$

where $S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$ $S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$

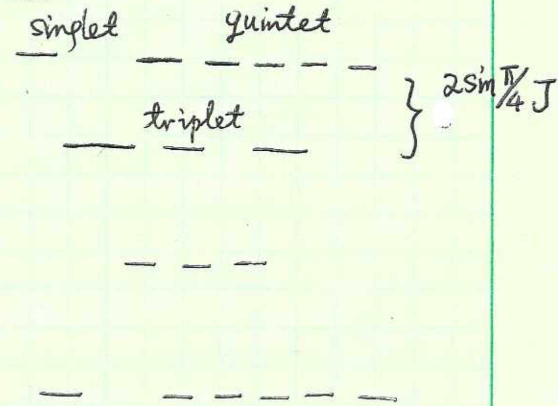
if $\theta = 0$, then it's the spin-1 Heisenberg model — 1D Haldan chain

		$\vec{S}_i \cdot \vec{S}_{i+1}$	$(\vec{S}_i \cdot \vec{S}_{i+1})^2$	E
SU(2)	singlet	-2	4	$-2\cos\theta + 4\sin\theta$
	triplet	-1	1	$-\cos\theta + \sin\theta$
	quintel	1	1	$\cos\theta + \sin\theta$

① At $\theta = \pi/4, \frac{3\pi}{4}$, we have $E_{singlet} = E_{quintet}$

$3 \times 3 = 6 + 3$ $\theta = \pi/4$

This energy split pattern $\theta = 3\pi/4$



matches $\boxed{\square \times \square = \begin{smallmatrix}\square\\ \square\end{smallmatrix} + \square\square}$

of SU(3)

Pick up a bond with sites i and $i+1$. On each site, we define state $|a\rangle$, $a = x, y, z$, satisfying $S_a |a\rangle = 0$.

Then $|a\rangle_i$ form a triplet representation of $SU(2)$, and also the fundamental of $SU(3)$.

$$S_z : \lambda_2, \quad S_x : \lambda_7, \quad S_y : -\lambda_5$$

$$\Rightarrow \vec{S}_i \cdot \vec{S}_{i+1} = \lambda_2(i)\lambda_2(i+1) + \lambda_7(i)\lambda_7(i+1) + \lambda_5(i)\lambda_5(i+1)$$

$$(\vec{S}_i \cdot \vec{S}_{i+1})^2 = \lambda_2^2(i)\lambda_2^2(i+1) + \lambda_7^2(i)\lambda_7^2(i+1) + \lambda_5^2(i)\lambda_5^2(i+1)$$
$$+ \lambda_2(i)\lambda_7(i)\lambda_2(i+1)\lambda_7(i+1) + \lambda_2(i)\lambda_5(i)\lambda_2(i+1)\lambda_5(i+1)$$
$$+ \lambda_5(i)\lambda_7(i)\lambda_5(i+1)\lambda_7(i+1) + (i \leftrightarrow i+1)$$

$$\lambda_2^2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} = \frac{2}{3} + \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \frac{2}{3} + \frac{1}{\sqrt{3}}\lambda_8$$

$$\lambda_7^2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \frac{2}{3} - \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \frac{2}{3} - \frac{\lambda_3}{2} - \frac{\sqrt{3}}{6}\lambda_8$$

$$\lambda_5^2 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 1 \end{pmatrix} = \frac{2}{3} + \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} = \frac{2}{3} + \frac{\lambda_3}{2} - \frac{\sqrt{3}}{6}\lambda_8$$

$$\lambda_2^2(i)\lambda_2^2(i+1) + 2\to 7 + 2\to 5$$
$$= \frac{4}{9}\times 3 + \frac{1}{3}\lambda_8(i)\lambda_8(i+1) + \frac{\lambda_3(i)\lambda_3(i+1)}{4}\times 2 + \frac{1}{12}\times 2\,\lambda_8(i)\lambda_8(i+1)$$
$$+ \frac{2}{3\sqrt{3}}\left[\lambda_8(i) + \lambda_8(i+1)\right] + \frac{2}{3}\Big[-\cancel{\frac{\lambda_3}{2}(i)} - \frac{\sqrt{3}}{6}\lambda_8(i) - \cancel{\frac{\lambda_3}{2}(i+1)} - \frac{\sqrt{3}}{6}\lambda_8(i+1)$$
$$+ \cancel{\frac{\lambda_3}{2}(i)} - \frac{\sqrt{3}}{6}\lambda_8(i) + \cancel{\frac{\lambda_3}{2}(i+1)} - \frac{\sqrt{3}}{6}\lambda_8(i+1)\Big]$$

$$= \frac{4}{3} + \frac{1}{2}\left[\lambda_3(i)\lambda_3(i+1) + \lambda_8(i)\lambda_8(i+1)\right]$$

$$\lambda_2\lambda_7 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & & -1 \\ & & \\ & & \end{pmatrix} = \frac{-1}{2}\begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} - \frac{i}{2}\begin{bmatrix} & & -i \\ & & \\ i & & \end{bmatrix}$$

$$= -\frac{1}{2}\lambda_4 - \frac{i}{2}\lambda_5$$

$$\lambda_7\lambda_2 = \begin{pmatrix} 0 & & \\ & & -i \\ & i & \end{pmatrix}\begin{pmatrix} & -i & \\ i & & \\ & & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ -1 & & \end{pmatrix} = -\frac{\lambda_4}{2} + \frac{i}{2}\lambda_5$$

hence
$$\lambda_2(i)\lambda_7(i)\,\lambda_2(i+1)\lambda_7(i+1) + (i\leftrightarrow i+1) = \left(-\frac{\lambda_4}{2}(i) - \frac{i}{2}\lambda_5(i)\right)\left(-\frac{\lambda_4}{2}(i+1) - \frac{i}{2}\lambda_5(i+1)\right)$$
$$+\left(-\frac{\lambda_4}{2}(i) + \frac{i}{2}\lambda_5(i)\right)\left(-\frac{\lambda_4}{2}(i+1) + \frac{i}{2}\lambda_5(i+1)\right)$$

$$= 2\times\frac{1}{4}\lambda_4(i)\,\lambda_4(i+1) - 2\times 1/4\,\lambda_5(i)\lambda_5(i+1) = \frac{1}{2}\left[\lambda_4(i)\lambda_4(i+1) - \lambda_5(i)\lambda_5(i+1)\right]$$

$$\lambda_2\lambda_5 = \begin{pmatrix} 0 & -i & \\ i & & \\ & & 0 \end{pmatrix}\begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix} = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & & \\ & & -i \\ & i & 0 \end{pmatrix} = \frac{\lambda_6}{2} + \frac{i}{2}\lambda_7$$

$$\lambda_5\lambda_2 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}\begin{pmatrix} 0 & -i & \\ i & & \\ & & 0 \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix} = \frac{\lambda_6}{2} - \frac{i}{2}\lambda_7$$

$$\Rightarrow \lambda_2(i)\lambda_5(i)\,\lambda_2(i+1)\lambda_5(i+1) + (i\leftrightarrow i+1) = \frac{1}{2}\left[\lambda_6(i)\lambda_6(i+1) - \lambda_7(i)\lambda_7(i+1)\right]$$

$$\lambda_5\lambda_7 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}\begin{pmatrix} & & \\ & & -i \\ & i & \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ & & \\ & & \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{\lambda_1}{2} + \frac{i\lambda_2}{2}$$

$$\lambda_7\lambda_5 = (\lambda_5\lambda_7)^\dagger = \frac{\lambda_1}{2} - \frac{i\lambda_2}{2}$$

$$\Rightarrow \lambda_5(i)\lambda_7(i)\,\lambda_5(i+1)\lambda_7(i+1) + (i\leftrightarrow i+1) = \frac{1}{2}\left[\lambda_1(i)\lambda_1(i+1) - \lambda_2(i)\lambda_2(i+1)\right]$$

$$\Rightarrow \left(\vec{S}_i\cdot\vec{S}_{i+1}\right)^2 = \frac{4}{3} + \frac{1}{2}\left[\sum_{a=1,3,4,6,8}\lambda_a(i)\,\lambda_a(i+1) - \sum_{a=2,5,7}\lambda_a(i)\lambda_a(i+1)\right]$$

$$\Rightarrow H = J\left(\underbrace{\frac{1}{2}\sin\theta\sum_{a=13468}\lambda_a(i)\lambda_a(i+1)}_{\text{real part}} + \underbrace{\left(\cos\theta - \frac{\sin\theta}{2}\right)\sum_{a=25\,7}\lambda_a(i)\lambda_a(i+1)}_{\text{imaginary matrices}}\right)$$

Apparently at $\sin\theta = \cos\theta$.

$$H = \pm \frac{\sqrt{2}}{4} J \sum_{a=1}^{8} \lambda_a(i)\, \lambda_a(i+1).$$

anti-symmetric rank-2 tensor

C-G decomposition: $|a\rangle_i |b\rangle_{i+1} \xrightarrow{P_0} \frac{1}{\sqrt{2}} \left[\epsilon^{abc} |b\rangle_i |c\rangle_{i+1}\right]$.

symmetric rank-2 tensor

$$\begin{cases} |a\rangle_i |a\rangle_{i+1} \\ \frac{1}{\sqrt{2}}\left[|a\rangle_i |b\rangle_{i+1} + |b\rangle_i |a\rangle_{i+1}\right] \quad (a \neq b) \end{cases}$$

s: $\frac{1}{\sqrt{3}}\left(|x\rangle_i |x\rangle_i + |y\rangle_i |y\rangle_{i+1} + |z\rangle_i |z\rangle_{i+1}\right)$

d: $\frac{1}{\sqrt{2}}\left(|x\rangle_i |x\rangle_{i+1} - |y\rangle_i |y\rangle_{i+1}\right)$, $\frac{1}{\sqrt{6}}\left(|x\rangle_i |x\rangle_{i+1} + |y\rangle_i |y\rangle_{i+1} - 2|z\rangle_i |z\rangle_{i+1}\right)$, $\frac{1}{\sqrt{2}}\left[|x\rangle_i |y\rangle_{i+1} + |y\rangle_i |x\rangle_{i+1}\right]$, $\frac{1}{\sqrt{2}}\left[|y\rangle_i |z\rangle_{i+1} + |z\rangle_i |y\rangle_{i+1}\right]$, $\frac{1}{\sqrt{2}}\left[|z\rangle_i |x\rangle_{i+1} + |x\rangle_i |z\rangle_{i+1}\right]$

$$\square \times \square \to \begin{smallmatrix}\square\\ \square\end{smallmatrix} + \square\square$$

$$= \square^* + \square\square$$

HW: Please verify that for the 2-site triplet

$$|\psi^a\rangle_{i\,i+1} = \frac{1}{\sqrt{2}} \epsilon^{abc}\left(|b\rangle_i |c\rangle_{i+1}\right).$$

They transform according to the [illegible] anti-fundamental Rep of SU(3).

There are another $SU(3)$ symmetric point at $\cos\theta = 0$, i.e. $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$.

then $$H = \pm J/2 \left[\sum_{a=1,3,4,6,8} \lambda_a(i)\lambda_a(i+1) - \sum_{a=2,5,7} \lambda_a(i)\lambda_a(i+1) \right]$$

consider on even lattice sites we choose the fundamental Rep $\square$, with λ_a

on odd lattice sites, we choose the anti-fundamental Rep $\square^*$, such that

$$\lambda'_a = -\lambda_a \quad \text{for } a = 13468 \text{ (real)}, \qquad \lambda'_a = \lambda_a \quad \text{for } a = 2,5,7$$

Then we have

$$\boxed{H = \mp J/2 \sum_{a=1\sim 8} \lambda_a(i)\lambda'_a(i+1) .}$$

under $SU(3)$ transformations ($\square$ for $i \in$ even, $\square^*$ for $i \in$ odd)

in both cases
$$\left.\begin{aligned} \mathcal{U}(R)\lambda_a \mathcal{U}(R^{-1}) &= \sum \lambda_b D^{ad}_{ba}(R) \\ \mathcal{U}^*(R)\lambda'_a \mathcal{U}^*(R^{-1}) &= \sum \lambda'_b D^{ad}_{ba}(R) \end{aligned}\right\}$$

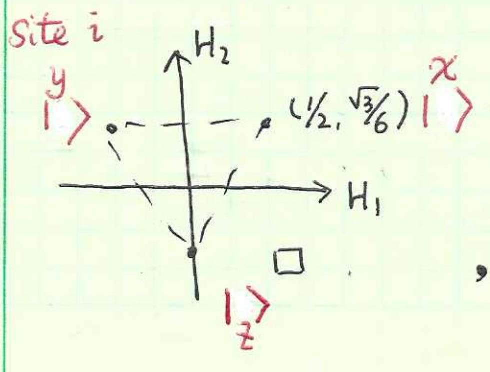
$\Rightarrow$ H is invariant under the staggered $SU(3)$ transformations.

At $\cos\theta = 0$, we have degeneracy between triplet and quintet.

i.e.

$\square \times \square^* = \bullet + $ (adjoint Young diagram) or $3 \times 3 = 1 + 8$

singlet $\qquad$ quintet + triplet



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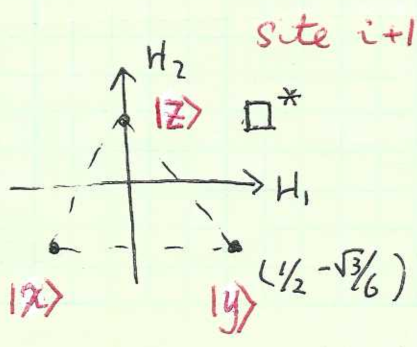


the SU(3) for two-sites . is generated for $\lambda_a + \lambda_a'$

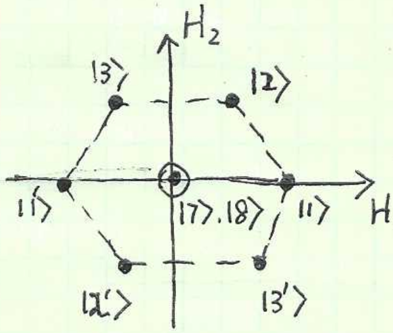
or $\exp[i\theta_a(\lambda_a + \lambda_a')]$.

The singlet is $\boxed{\frac{1}{\sqrt{3}}\{|x\rangle_i|x\rangle_{i+1} + |y\rangle_i|y\rangle_{i+1} + |z\rangle_i|z\rangle_{i+1}\}}$

The other 8 - states :

$$\lambda_3^{tot} = \lambda_3 + \lambda_3' = \lambda_3(i) - \lambda_3(i+1)$$

$$\lambda_8^{tot} = \lambda_8(i) - \lambda_8(i+1)$$



hence for states

$$|7\rangle = \frac{1}{\sqrt{2}}[|x\rangle_i|x\rangle_{i+1} - |y\rangle_i|y\rangle_{i+1}]$$

$$|8\rangle = \frac{1}{\sqrt{6}}\{|x\rangle_i|x\rangle_{i+1} + |y\rangle_i|y\rangle_{i+1} - 2|z\rangle_i|z\rangle_{i+1}\}$$

their $(\lambda_3^{tot}, \lambda_8^{tot}) = (0, 0)$

$|1\rangle = |x\rangle_i|y\rangle_{i+1} \longrightarrow (1/2, \frac{\sqrt{3}}{6}) + (1/2, -\sqrt{3}/6) = (1, 0)$

$|1'\rangle = |y\rangle_i|x\rangle_{i+1} \longrightarrow (-1/2, \sqrt{3}/6) + (-1/2, -\sqrt{3}/6) = (-1, 0)$

$|2\rangle = |x\rangle_i|z\rangle_{i+1} \rightarrow (1/2, \sqrt{3}/6) + (0, \sqrt{3}/3) = (1/2, \frac{\sqrt{3}}{2})$

$|2'\rangle = |z\rangle_i|x\rangle_{i+1} \rightarrow (0, -\sqrt{3}/3) + (-1/2, -\sqrt{3}/6) = (-1/2, -\sqrt{3}/2)$

$|3\rangle = |y\rangle_i|z\rangle_{i+1} \rightarrow (-1/2, \sqrt{3}/6) + (0, \frac{\sqrt{3}}{3}) = (-1/2, \frac{\sqrt{3}}{2})$

$|3\rangle' = |z\rangle_i|y\rangle_{i+1} \rightarrow (0, -\frac{\sqrt{3}}{3}) + (1/2, -\sqrt{3}/6) = (1/2, -\frac{\sqrt{3}}{2})$

They are octet — meson states $q\bar{q}$

Phase diagram of the SU(3) (spin-1 chain)

① FM region $\frac{5\pi}{4} \geq \theta \geq \frac{\pi}{2}$, we have $E_{quint} \leq E_{singlet}$, $(\theta = 5\pi/4)$

$E_{quintet} \leq E_{triplet}$ $(\theta = \pi/2)$

② SU(3) line: $\theta = \pi/4$, we have $E_{quint} = E_{singlet}$

at $\pi/4 < \theta < \pi/2$, we have $E_{triplet} < E_{quintet} < E_{singlet}$

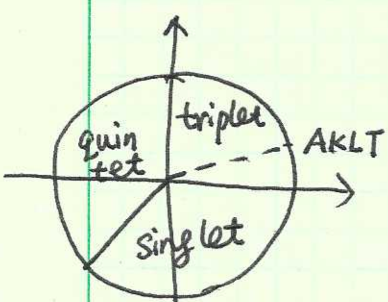
this is the gapless phase

③ Between $-\pi/4 < \theta < \pi/4$, it's the Halden phase – spin gapped

In this region, E_{quinte} is the largest, and the low energy sectors are the singlet and triplet. $E_{singlet} = E_{triplet}$ at $\theta = \tan^{-1} 1/3$, which is the AKLT point. When $\theta \geq \tan^{-1} 1/3$, we have

$E_{triplet} \leq E_{singlet} \leq E_{quintet}$.

when $\theta \leq \tan^{-1} 1/3$, we have $E_{singlet} \leq E_{triplet} \leq E_{quintet}$

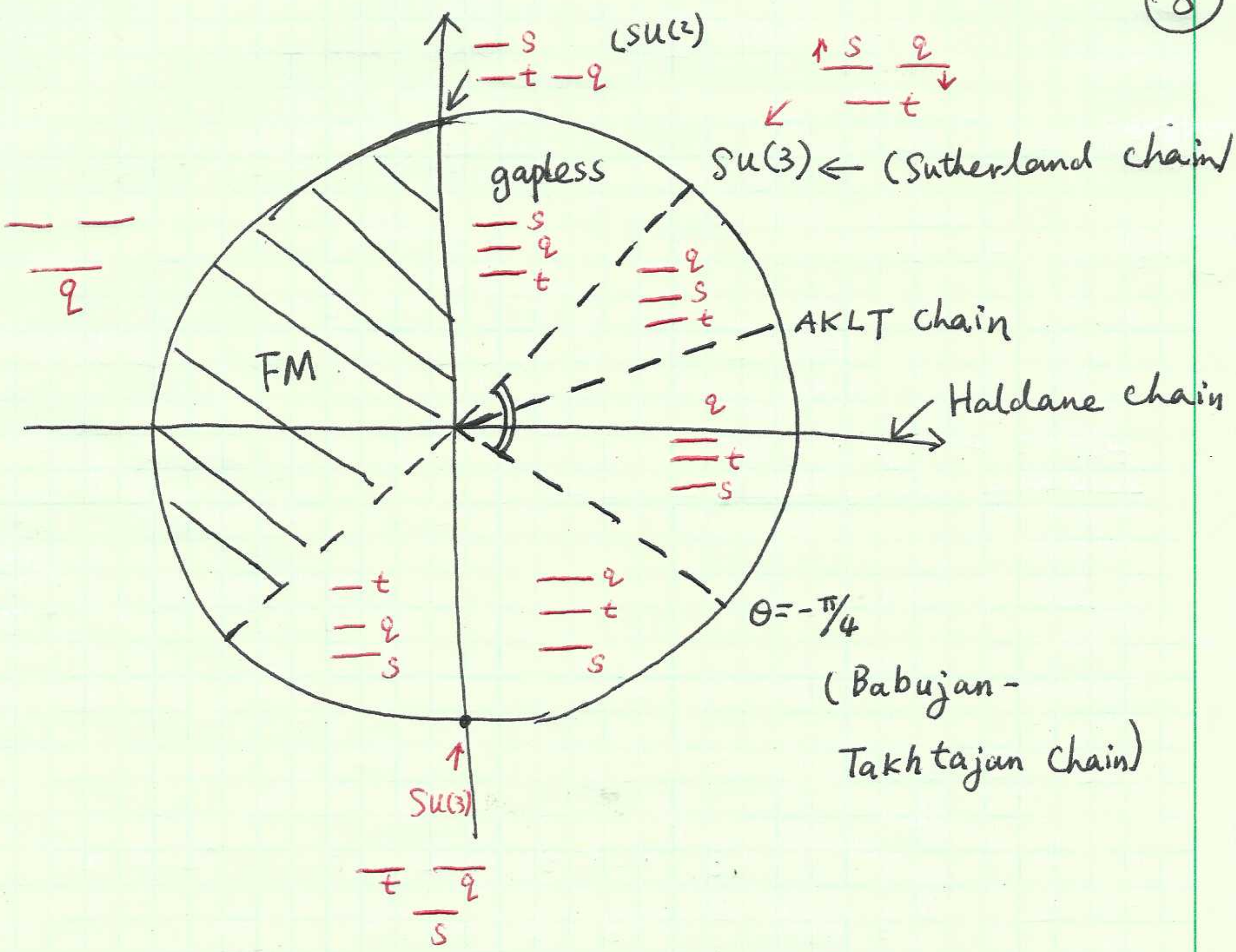


④ $\theta = -\pi/4$ is an exact Solvable point (Babujan – Takhtajan chain) it's a gapless phase

⑤ $\theta \in [-\pi/4, -3\pi/4] \rightarrow$ dimerized phase

The staggered SU(3) lies in $\theta = -\pi/2$

inwhich the $E_{singlet} < E_{triplet} = E_{quintet}$


s
t
q
(SU(2)
s
q
t
gapless
SU(3) ← (Sutherland chain)
q
FM
q
s
t
AKLT chain
Haldane chain
q
t
s
t
q
s
q
t
s
θ=-π/4
(Babujan-
Takhtajan Chain)
SU(3)
t
q
s