The scaling hypothesis

The critical exponents defined before are not completely independent. They satisfy the following laws

Fisher	$\gamma = \nu(2-\eta)$,	Rushbrooke	$\alpha + 2\beta + \gamma = 2$,
widom	$\gamma = \beta(\delta - 1)$,	Josephson	$\nu d = 2 - \alpha$.

§ Widom scaling theory

define dimensionless variables $t = \frac{T - T_c}{T_c}$ and $h = \frac{\mu B}{k_B T_r}$

then the free energy density can be represented as $f = f(t, h)$. f can be decomposed into a regular part and a singular part as

$f = f_n + f_s$, and f_s-part is responsible for critical phenomenon.

Widom assume f_s can be represented as homogenous function of t, h as

$$\boxed{f[\lambda^p t, \lambda^q h] = \lambda f[t, h]}$$

p and q are scaling powers. depend on universal classes

From p, q, we can derive all the critical exponents

① $m \sim -\left(\frac{\partial f}{\partial h}\right)_t$. we $\lambda^q \frac{\partial}{\partial(\lambda^q h)} f(\lambda^p t, \lambda^q h) = \lambda \frac{\partial}{\partial h} f(t, h)$

$$\lambda^q m(\lambda^p t, \lambda^q h) = \lambda m(t, h).$$

set $\lambda^p t = -1$, $h = 0 \Rightarrow$ $m(t, 0) = \lambda^{q-1} m(-1, 0)$.

From $\lambda^p t = -1 \Rightarrow \lambda = (-t)^{-1/p} \Rightarrow m(t, 0) \sim (-t)^{(1-q)/p} \Rightarrow \boxed{\beta = \frac{1-q}{p}}$

Comment: ① we can set $\lambda^p t = 1$, but $m(t,0) = \lambda^{q-1} m(1,0) = 0$
nothing interesting.

② rigorously speaking, the relation $f(\lambda^p t, \lambda^q h) = \lambda f(t,h)$ only valid in the regime of $|t| \ll 1$, $h \ll 1$. We cannot set $\lambda^p t = -1$. In fact we set $\lambda^p t \sim t_m$, where t_m is the size of the critical region. This doesnot change the scaling behavior. Setting $\lambda^p t = -1$ is only a convenience

② set $t=0$, $\lambda^q h = 1$; we have

$$\lambda^{q-1} m(0,1) = m(0,h) \Rightarrow m(0,h) = h^{\frac{q-1}{q}} m(0,1) \Rightarrow \boxed{\delta = \frac{q}{1-q}}$$

③ From $\lambda^q m(\lambda^p t, \lambda^q h) = \lambda m(t,h)$, we take derivative with respect to h.

$$\Rightarrow \lambda^{q+q} \chi(\lambda^p t, \lambda^q h) = \lambda \chi(t,h) \quad \text{or} \quad \chi(t,h) = \lambda^{2q-1} \chi(\lambda^p t, \lambda^q h)$$

set $\begin{cases} \lambda^p t = \pm 1 & \text{for } t>0, \text{ and } t<0 \\ h=0 \end{cases}$

$$\Rightarrow \chi(t,0) = |t|^{-\frac{2q-1}{p}} \chi(\pm 1, 0) \sim |t|^{-\frac{2q-1}{p}} \Rightarrow \boxed{\gamma = \frac{2q-1}{p}}$$

④ from $f[\lambda^p t, \lambda^q h] = \lambda f(t,h)$

$$\lambda^p \frac{\partial}{\partial(\lambda^p t)} f[\lambda^p t, \lambda^q h] = \lambda \frac{\partial}{\partial t} f(t,h] \Rightarrow \lambda^{2p} \frac{\partial^2 f}{\partial[\lambda^p t]^2}(\lambda^p t, \lambda^q h) = \lambda \frac{\partial}{\partial t} f(t,h]$$

$$C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_h \sim \frac{\partial^2 f}{\partial t^2}. \qquad \Rightarrow \lambda^{2p} C[\lambda^p t, \lambda^q h] = \lambda C(t,h]$$

set $h=0$, $\lambda^p t = \pm 1$ $\Rightarrow$ $C(t,0) = |t|^{\frac{2p-1}{p}} C[\pm 1, 0]$ for $t>0$, $t<0$ respectively.

$$\Rightarrow \boxed{\alpha = \frac{2p-1}{p}}$$

we can check that $\alpha + 2\beta + \gamma = \frac{2p-1}{p} + \frac{2(1-q)}{p} + \frac{2q-1}{p} = 2$

$$\gamma = \beta(\delta - 1) : \quad \frac{2q-1}{p} = \frac{1-q}{p}\left[\frac{q}{1-q} - 1\right]$$

§ Fisher scaling law

In order to find the relations of η, ν with $\alpha \beta, \gamma, \delta$, we need consider correlation function $G(r) \xrightarrow[t \to 0^{\pm}]{} |r|^{-(d-2+\eta)} g_{\pm}(r/\xi)$ at $h=0$.

at $h \neq 0$ $\quad G(r,h) \xrightarrow[t \to 0^{\pm}]{} |r|^{-(d-2+\eta)} g'_{\pm}(r/\xi, h\xi^{y})$.

y is a const power. The magnetic moments within a correlation length ξ are nearly the same. Thus the effect of external field h, is amplified by a factor of ξ^{y}. Replacing $\xi \sim |t|^{-\nu}$, we have

$$\boxed{G(r,h) \xrightarrow[t \to 0^{\pm}]{} |r|^{-(d-2+\eta)} g'_{\pm}(r/\xi, h/|t|^{\Delta})}, \text{ with } \Delta = \nu y.$$

① $h=0$. for Fisher's law.

$$\chi = \frac{1}{N k_B T} \sum_{i,j} [\langle m(i)\, m(j) \rangle - \langle m(i) \rangle \langle m(j) \rangle] \longleftarrow \text{please prove}$$

$$= \frac{1}{k_B T} \int d^d r\, G(r) \Big|_{h=0} \sim \int d^d r \; r^{-(d-2+\eta)} g_{\pm}(r/\xi)$$

$$\sim \left[\int_0^{\infty} dx \; x^{-(d-2+\eta)} g_{\pm}(x)\right] \cdot \xi^{2-\eta} = \text{const} \cdot \xi^{2-\eta}$$

$$\Rightarrow \quad \chi \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}$$

and $\chi \sim |t|^{-\gamma}$

$$\Big\} \Rightarrow \boxed{\gamma = \nu(2-\eta)}$$

The assumption that ξ is the only relevant length scale near $t \sim 0$ is called "scaling hypothesis." There's another "hyperscaling hypothesis" which is necessary to derive the Josephson's law.

hyperscaling hypothesis : The singular part of the free energy density scales $f_{sing}(t) \sim \dfrac{1}{\xi(t)^d} \sim |t|^{\nu d}$.

From this assumption $C \sim \dfrac{\partial^2 f_{sing}}{\partial t^2} \sim |t|^{\nu d - 2}$, and $C \sim |t|^{-\alpha}$

$$\Rightarrow \quad \boxed{\alpha = 2 - \nu d.}$$

Argument to justify "hyper-hypothesis": Pippard & Ginsberg. At tempertature $k_B T \sim k_B T_c$, the fluctuation length scale is ξ. The free energy deviation from the mean field within ξ, is at the order of $k_B T_c$, thus $\beta \Delta F \sim 1$. Thus $f \sim \dfrac{k_B T_c}{\xi^d} \sim |t|^{\nu d}$.

In summary, from the above 4-scaling laws, we can express all the critical exponents in terms of η and ν as.

η: Critical behavior of two point correlation

ν: divergence of correlation length

$$\boxed{\begin{aligned} \alpha &= 2 - \nu d \\ \beta &= \tfrac{1}{2}\nu(d-2+\eta) \\ \gamma &= \nu(2-\eta) \\ \delta &= \frac{d+2-\eta}{d-2+\eta} \end{aligned}}$$

The goal of RG is to justify these scaling argument and provides a technical frame work to compute these critical exponents.

	2d Ising	3d Ising	3D Heinsenberg	mean field
α	0	0.12	-0.14	0
β	1/8	0.31	0.3	1/2
γ	7/4	1.25	1.4	1
δ	15	5		3
ν	1	0.64	0.7	1/2
η	1/4	0.05	0.04	0

- mean-field results do not obey Josephson law: $2-\alpha/\nu d = 4/d$.