4 - ε expansion: beyond Gaussian

We now consider interaction term $u_0 \neq 0$. In momentum space

$$F = F_0 + \frac{1}{4} u_0 \int_{\Lambda} \frac{dk_1 \cdots dk_4}{(2\pi)^{4d}} S(k_1) S(k_2) S(k_3) S(k_4) (2\pi)^4 \delta(k_1+k_2+k_3+k_4)$$

we separate fast and slow modes; they couple in $F_I(S_>, S_<)$:

$$\boxed{S(k) = S_>(k)\, \theta(\Lambda > k > \Lambda/\ell) + S_<(k)\, \theta(\Lambda/\ell > k)}$$

then $$Z = \int DS_< \, e^{-F_0(S_<)} \int DS_> \, e^{-F_0(S_>) - F_I(S_>, S_<)}$$

$$= \text{const} \cdot \int DS_< \, e^{-F_0(S_<)} \langle e^{-F_I(S_>, S_<)} \rangle_0$$

where $$\langle e^{-F_I(S_<, S_>)} \rangle_0 \equiv \frac{\int DS_> \, e^{-F_0(S_>)} \, e^{-F_I(S_>, S_<)}}{\int DS_> \, e^{-F_0(S_>)}}$$

partial trace average over fast modes

Since F_0 is Gassian, we have the cummulate expansion

$$\langle e^{-F_I} \rangle_0 = \exp\left[-\left\{ \langle F_I \rangle_0 - \frac{1}{2} \left(\langle F_I^2 \rangle_0 - (\langle F_I \rangle_0)^2 \right) \right\} + \cdots \right]$$

$\Rightarrow$ $Z = \int D(S_<) \exp\{-F(S_<)\}$, with

$$F(S_<) = F_0(S_<) + \langle F_I \rangle_0 - \frac{1}{2} \left[\langle F_I^2 \rangle_0 - (\langle F_I \rangle_0)^2 \right] + \cdots$$

$$F_0(S_<) = \int_{\Lambda/\lambda} \frac{dk}{(2\pi)^d} \frac{1}{2} (k^2 + r) \, |S_<(k)|^2$$

① First order $\langle F_I(S_>, S_<)\rangle_c$

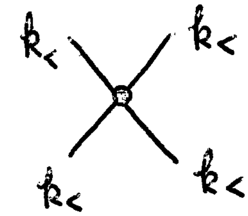
$$F_I = \frac{u_0}{4}\int_\Lambda \frac{dk_1\cdots dk_4}{(2\pi)^{4d}}\,(2\pi)^d\,\delta(k_1+k_2+k_3+k_4)\left[S_>(k_1)+S_<(k_1)\right]\cdots\left[S_>(k_4)+S_<(k_4)\right]$$

$$\Rightarrow S(k_1)S(k_2)S(k_3)S(k_4) = \underbrace{S_>S_>S_>S_>}_{\text{constant}} + 4\underbrace{S_>S_>S_>S_<}_{0} + 4\underbrace{S_<S_<S_<S_>}_{0}$$

$$+\,6\underbrace{S_>S_>S_<S_<}_{\text{one loop}} + \underbrace{S_<S_<S_<S_<}_{\text{tree level}}$$

tree level:

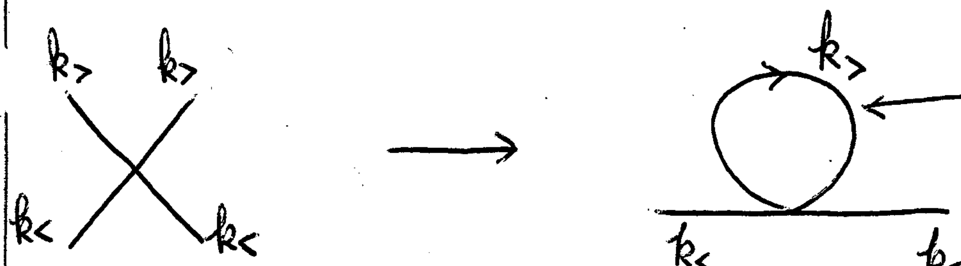
$$\Delta F^{\text{tree}} = \frac{u_0}{4}\int_{k<\Lambda/\ell}\frac{d^dk_1\cdots d^dk_4}{(2\pi)^{4d}}\,S_<(k_1)S_<(k_2)S_<(k_3)S_<(k_4)\cdot(2\pi)^d\,\delta(k_1+k_2+k_3+k_4)$$



one-loop:

$$\Delta F^{(1)} = \frac{u_0}{4}\cdot 6\int_0^{\Lambda/\ell}\frac{dk_1 dk_2}{(2\pi)^{2d}}\,S_<(k_1)\,S_<(k_2)\,(2\pi)^d\,\delta(k_1+k_2)$$

$$\int_{\Lambda/\ell}^{\Lambda}\frac{dk_3}{(2\pi)^d}\,\frac{1}{r_0+k_3^2}$$



integrate only over fast modes

$$\boxed{\langle S_>(k_3)\,S_>(k_4)\rangle_0 = (2\pi)^d\,\delta(k_3+k_4)\,\frac{1}{k_3^2+r_0}}$$

Ex: please verify it.

define integral $I_1 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{(2\pi)^d} \frac{1}{k^2 + r_0}$

$$\Rightarrow \quad \Delta F^{(1)} = I_1 \frac{3}{2} u_0 \int_{\Lambda/\ell} \frac{dk}{(2\pi)^d} |S_<(k)|^2$$

Add everything ~~together~~ at 1st order of u, we have

$$F(S_<) = \int_{\Lambda/\ell} \frac{dk}{(2\pi)^d} \frac{1}{2}\left[k^2 + r_0 + 3u_0 I_1(\Lambda,\ell)\right] |S_<(k)|^2$$

$$+ \frac{u_0}{4} \int_{\Lambda/\ell} S_<(k_1) S_<(k_2) S_<(k_3) S_<(k_4) (2\pi)^d \delta(k_1+k_2+k_3+k_4)$$

$$\boxed{\text{define } k' \equiv \ell k, \text{ and } S'(k') = \ell^{-d/2-1} S_<(k/\ell)}$$

$$\Rightarrow F(S_<) = \int \frac{dk'}{(2\pi)^d} \frac{1}{2}\left[k'^2 + \ell^2 (r_0 + 3u_0 I_1(\Lambda,\ell))\right] |S'(k')|^2$$

$$+ \frac{u_0}{4} \ell^{-3d} \ell^{(d/2+1)\cdot 4} \int \frac{dk_1' \cdots dk_4'}{(2\pi)^{4d}} S'(k_1') \cdots S'(k_4') (2\pi)^d \delta(k_1'+k_2'+k_3'+k_4')$$

Thus to 1rst order, we have

$$\boxed{\begin{aligned} r(\ell) &= \ell^2 (r_0 + 3u_0 I_1(\Lambda,\ell)] \\ u(\ell) &= \ell^{\epsilon} u_0, \qquad \text{where } \epsilon = 4-d \end{aligned}}$$

we will evaluate $I_1(\Lambda,\ell)$ later.

what does $u(\ell) = \ell^{\epsilon} u_0$ mean? The tree level results just reflects that u_0 carries dimension. Just like that

r_0 carries dimension L^{-2}, thus we express

$r_0 = \boxed{r_0'}\, a^{-2}$ where r_0' is dimensionless, $a = 1/\Lambda$ is the length unit. Where we do coarse averging, we change to a ~~d~~ larger length unit λa, thus $r_0 = \boxed{r_0' \lambda^2}\, (\lambda a)^{-2}$ unit,

Thus a nother way to under RG: we are changing the unit of length (using a longer and longer ruler). Relative to the new unit, the value of physical quantity changes. This is just the meaning of naive dimension, which usually are tree-level contributions of RG. Similarly, we have know before that u_0 carries the unit $L^{-\epsilon}$, thus we write $u_0 = u_0' a^{-\epsilon} = (u_0' \, l^{\epsilon})(la)^{-\epsilon}$.

The one loop integral gives additional corrections from fluctuation, which will change the naive scalling. We have seen the correction at loop level to r, but for u, we only have tree level result yet. To be consistent, we need also calculate the fluctuation correction to the scaling of u, i.e. one loop level!

We need some knowledge of Gaussian integral. For details, please refer to Goldenfeld Shap 12, Sect 12.3.3.

§ Gaussian integral

With respect to the Gaussian weight $F_0 = \frac{1}{2}\int_{\Lambda/\ell}^{\Lambda} \frac{dk}{(2\pi)^d} (r_0 + k^2)|S_>(k)|^2$

or the discrete version $F_0 = \frac{1}{V}\sum_k{}' \frac{1}{2}(r_0 + k^2)\,|S_>(k)|^2$, we have

① $$\langle S_>(k_1) S_>(k_2)\rangle = \frac{\int DS_>\ S_>(k_1) S_>(k_2)\, e^{-F_0(S_>)}}{\int DS_>\ e^{-F_0(S_>)}}$$

$$= \begin{cases} \delta_{k_1+k_2,0}\cdot V\cdot G_0(k) \\ (2\pi)^d\,\delta(k_1+k_2)\ G_0(k) \quad (\text{continuum limit}) \end{cases}$$

with $G_0(k) \equiv \dfrac{1}{k^2 + r_0}$ (please prove it!)

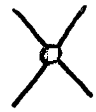
② $\langle S_>(k_1) S_>(k_2) \cdots S_>(k_m)\rangle = 0$ if m is odd.

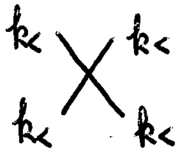
③ $$\langle S_>(k_1) S_>(k_2) S_>(k_3) S_>(k_4)\rangle = (2\pi)^d\delta(k_1+k_2)G_0(k_1)(2\pi)^d\delta(k_3+k_4)G_0(k_3)$$
$$+ (2\pi)^d\delta(k_1+k_3)G_0(k_1)(2\pi)^d\delta(k_2+k_4)G_0(k_2)$$
$$+ (2\pi)^d\delta(k_1+k_4)G_0(k_1)(2\pi)^d\delta(k_2+k_3)G_0(k_3)$$

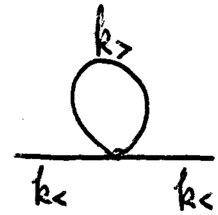
— Wick theorem

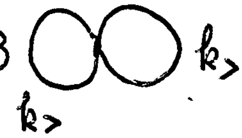
④ in general, we need to consider all the possible contractions

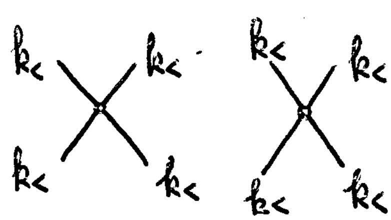
Now, 2-order calculation $\langle F_I^2\rangle_0 - \langle F_I\rangle_0^2$

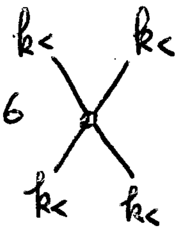
F_I : $\frac{u_0}{4}$; 

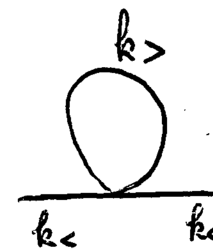
$\langle F_I\rangle_0 =$ 

$+ 6$ 

$+ 3$ 

$\langle F_I^2\rangle_0 =$ 

$+ 6$ 



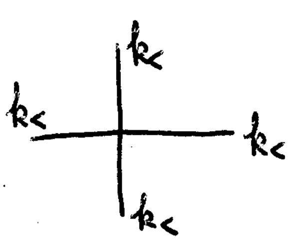
$+ \cdots$

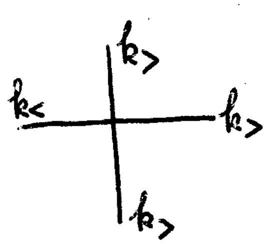
disconnected diagrams canceled by $\langle F_I\rangle_0^2$

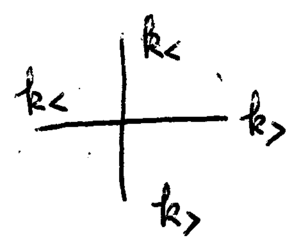
+ connected diagrams

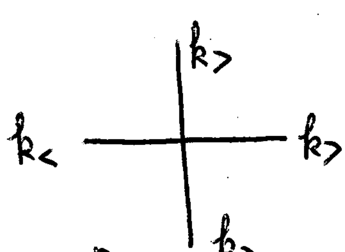
$\Rightarrow \langle F_I^2\rangle_0 - \langle F_I\rangle_0^2$: connected diagrames

Let us enumerate / ~~construct~~ connected diagrames

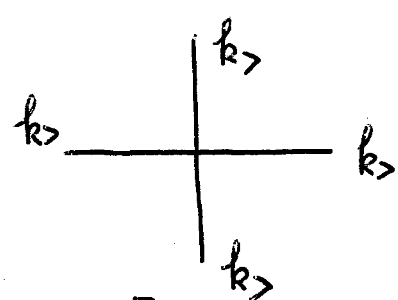
A 

B 

C 



D

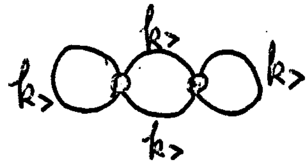


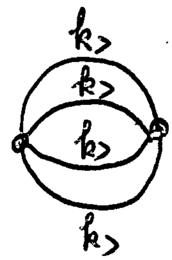
E

vertex of F_I

Connected diagrams

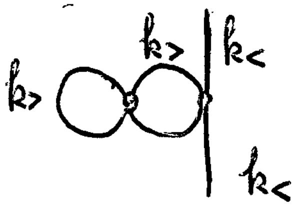
E-E :





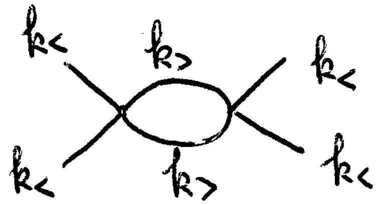
Constants : no contribution to $F(S_<)$.

E-C



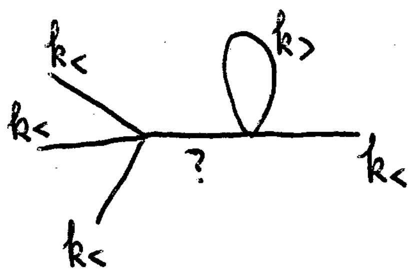
two-loop integral for r_0

C-C



one-loop for u

D-B

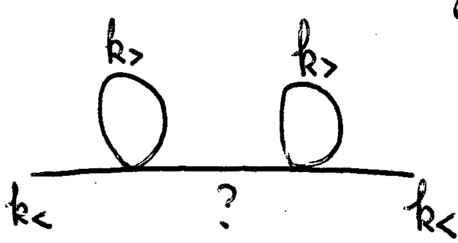


the - - - diagram is zero.

the line of "?" is internal link thus its mementum should be $k_>$. then it cannot match mementum conservation!

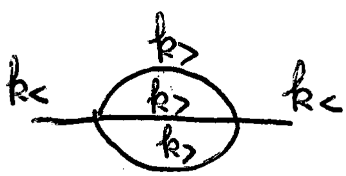
Similarly

D-D



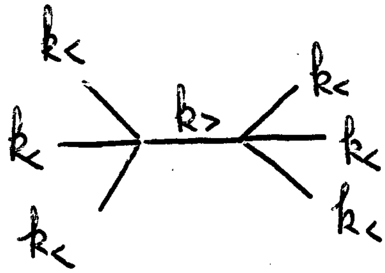
this diagram should also vanish

D-D



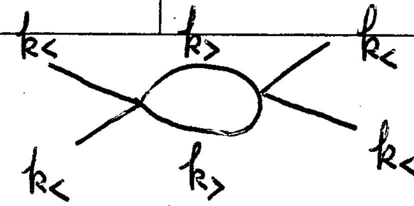
actually, this diagram is important. it gives rise to wavefunction renormalization because it depends on $k_<$

B-B



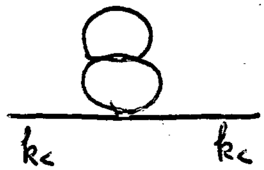
$\sim u^6$ irrelevant!

At one loop level, we only need

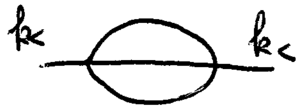


which renormalizes u.

At two loop level

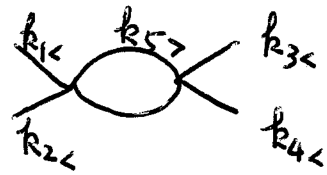


another $k_<$ independent correction to r_0



$k_<$ dependent, change coeffecient of $k_<^2$ term.

Let us only consider — one-loop:



$$-\frac{1}{2}\left(\frac{u}{4}\right)^2 \cdot \underbrace{\left(\frac{4!}{2!2!}\right)^2 \cdot 2}_{\text{sym-factor}}$$

① pick up 2 $k_<$ and $k_>$ for each vertex

② 2 different way for contraction

$$= -\frac{9u^2}{4}$$

$$\Rightarrow -\frac{9u^2}{4}\int_0^{\Lambda/\ell}\frac{dk_1\cdots dk_4}{(2\pi)^d}(2\pi)^d\delta(k_1+k_2+k_3+k_4)\, S(k_1)S(k_2)\cdots S(k_4)\int_{\Lambda/\ell}^{\Lambda/\ell}\frac{dk_5}{(2\pi)^d}\frac{1}{r_0+k_5^2}\frac{1}{r_0+(k_1+k_2-k_5)^2}$$

Thus the correction to u: (add tree level)

$$\frac{u_0}{4}\left[1-9u\int_0^{\Lambda/\ell}\frac{d^dk_5}{(2\pi)^d}\frac{1}{r_0+k_5^2}\frac{1}{r_0+(k_1+k_2-k_5)^2}\right]\int_0^{\Lambda/\ell}\frac{dk_1\cdots dk_4}{(2\pi)^d}$$

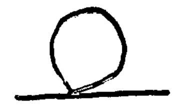
$$(2\pi)^d\delta(k_1+k_2+k_3+k_4)\,S(k_1)S(k_2)\cdots S(k_4)$$

$\rightarrow$ rescale $k' \equiv \ell k$, $S'(k') = z^{-1} S_<(k'/\ell)$ where $z = \ell^{d/2+1}$

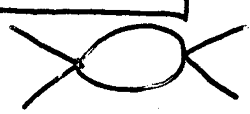
$$\Rightarrow u_0\, z^4 \cdot \ell^{-3d} \left(1 - \frac{9}{4} u\, I_2\right) = u(\ell)$$

Combine ~~together~~

momentum rescale



$$\boxed{\begin{aligned} k'^2 + r(\ell) &= z^2 \ell^{-d} \left(\frac{k'^2}{\ell^2} + r_0 + 3u_0\, I_1(\Lambda, \ell) \right] \\ u_0(\ell) &= z^4 \ell^{-3d}\, u_0 \left(1 - \tfrac{9}{4} u_0\, I_2(\Lambda, \ell, k) \right] \end{aligned}}$$



wavefun renormalization integral measure

because $I_1(\Lambda, \ell)$ has no dependence on the momentum of external leg.

we can just set $z = \ell^{-d/2+1}$ (naive scaling dimension)

$$\Rightarrow \boxed{\begin{aligned} r(\ell) &= \ell^2 \left(r_0 + 3u_0\, I_1(\Lambda, \ell) \right) \\ u(\ell) &= \ell^{\epsilon}\, u_0 \left(1 - 9 u_0\, I_2(\Lambda, \ell) \right) \end{aligned}}$$

In $I_2(\Lambda, \ell, k_<)$, we can set external integral to zero, otherwise we generate a momentum-dependent interaction, which is irrelevant and can be dropped.

Now we calculate
$$I_1 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{r_0 + k^2}$$

$$I_2 \equiv \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{(r_0 + k^2)^2}$$

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S_d: Solid angle in n-dim space, or, area of a unit sphere

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \qquad \text{check } S_2 = 2\pi$$
$$S_3 = \frac{2\pi^{3/2}}{\frac{1}{2}\pi^{1/2}} = 4\pi$$

① $$I_1 = \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{r_0 + k^2} = \frac{S_d \Lambda^{d-2}}{(2\pi)^d} \int_{1/\ell}^{1} \frac{dk' \cdot k'^{d-1}}{k_1'^2 + (r_0/\Lambda^2)}$$

denote $K_d = \frac{S_d}{(2\pi)^d}$ and $K_4 = \frac{2\pi^2}{(2\pi)^4} = \frac{1}{8\pi^2}$

The kneral $\int_{1/\ell}^{1} \frac{k'^{d-1}\, dk'}{k'^2 + r_0/\Lambda^2}$ we can set $d=4$, because around the fixed point, $u^* \propto \epsilon$

Thus $$\int_{1/\ell}^{1} \frac{\frac{1}{2} k'^2\, dk'^2}{k'^2 + r_0/\Lambda^2} = \frac{1}{2} \int_{1/\ell^2}^{1} \left(dy - \frac{r_0/\Lambda^2}{y + r_0/\Lambda^2}\, dy \right)$$

$$= \frac{1}{2} \left[\left(1 - 1/\ell^2\right) - \frac{r_0}{\Lambda^2} \ln \frac{1 + r_0/\Lambda^2}{1/\ell^2 + r_0/\Lambda^2} \right]$$

set $\ell^2 = 1 + 2\ln \ell$ and expand to $\ln \ell$ order (Taylor expansion)

$$\rightarrow \frac{1}{2} \left[2\ln \ell - \frac{r_0}{\Lambda^2} \frac{1}{1 + r_0/\Lambda^2} 2\ln \ell \right] = \ln \ell \frac{1}{1 + r_0/\Lambda^2}$$

$$\Rightarrow \boxed{I_1 = \Lambda^{d-2} \frac{K_4}{1 + r_0/\Lambda^2} \ln \ell}$$; in the Λ^{d-2} term, we did not set $d=4$, to maintain unit correct.

② $$I_2 = \int_{\Lambda/\ell}^{\Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{r_0 + k^2} \right)^2 = \frac{S_d \Lambda^{d-4}}{(2\pi)^d} \int_{1/\ell}^{1} \frac{k'^{d-1}\, dk'}{k_1'^2 + r_0/\Lambda^2}$$

$$\int_{1/l}^{1} \frac{k'^{3}\, dk'}{(k'^2 + r_0/\Lambda^2)^2} = \frac{1}{2}\int_{1/l}^{1} \frac{k'^2\, dk'^2}{(k'^2 + r_0/\Lambda^2)^2}$$

we have set $d=4$ because l^{ϵ} already contains a small ϵ.

$$= \frac{1}{2}\left[\int_{1/l}^{1} \frac{dk'^2}{k'^2 + r_0/\Lambda^2} - \int_{1/l}^{1} \frac{r_0/\Lambda^2\, dk'^2}{(k'^2 + r_0/\Lambda^2)^2}\right]$$

set $1/l = 1 - \ln l$

$$= \frac{1}{2}\left[\ln \frac{1 + r_0/\Lambda^2}{(1/l)^2 + r_0/\Lambda^2} - \frac{r_0}{\Lambda^2} \left.\frac{1}{k'^2 + r_0/\Lambda^2}\right|_{1/l}^{1}\right]$$

~~T~~ Take derivative with respect to $\ln l$, $\Rightarrow$

$$= \frac{1}{2}\cdot 2\ln l \left[\frac{1}{1 + r_0/\Lambda^2} - \frac{r_0}{\Lambda^2}\frac{1}{(1 + r_0/\Lambda^2)^2}\right] = \ln l \frac{1}{(1 + r_0/\Lambda^2)^2}$$

$$\Rightarrow \boxed{I_2 = \frac{\Lambda^{d-4} K_4}{(1 + r_0/\Lambda^2)^2}}$$

$d - 4 = -\epsilon$

$d - 2 = 2 - \epsilon$

Then the recursive relation:

$$r(l) = (1 + 2\ln l)\left[r_0 + \frac{3K_4 u_0 \Lambda^{2-\epsilon}}{(1 + r_0/\Lambda^2)} \ln l\right]$$

$$u(l) = (1 + \epsilon \ln l)\, u_0 \left[1 - \frac{9}{4}\frac{u_0 K_4 \Lambda^{-\epsilon}}{(1 + r_0/\Lambda^2)^2} \ln l\right]$$

$$\rightarrow \boxed{\begin{aligned} \frac{dr(l)}{d\ln l} &= 2r + \frac{A u \Lambda^{2-\epsilon}}{1 + r/\Lambda^2} \\ \frac{du}{d\ln l} &= u\left[\epsilon - \frac{B u \Lambda^{-\epsilon}}{(1 + r/\Lambda^2)^2}\right] \end{aligned}}$$

$A = 3K_4$

$B = 9\ K_4$

or, we can write in the dimensionless form

$$\begin{cases} \dfrac{d\, r(l)/\Lambda^2}{d \ln l} = 2\left(\dfrac{r}{\Lambda^2}\right) + A \dfrac{u \Lambda^{-\epsilon}}{1 + (r/\Lambda^2)} \\[2ex] \dfrac{d[u\Lambda^{-\epsilon}]}{d \ln l} = (u\Lambda^{-\epsilon}) \left[\epsilon - \dfrac{B u \Lambda^{-\epsilon}}{(1 + (r/\Lambda^2))^2}\right] \end{cases}$$