§1 $\boxed{\text{Theorem:}}$ The Hilbert space spanned by $|jm\rangle$ $(m=-j,\cdots j)$ is rotationally invariant and irreducible.

Proof: we denote such a space as L^j. Any vector in such a space can be expanded as $|\psi^j\rangle = \sum_m a_m |jm\rangle$. For any rotation $g(\hat{n},\theta)$, the associated rotation operator $D(g) = e^{-i\vec{J}\cdot\hat{n}\theta}$, we have

$$D(g)|\psi^j\rangle = \sum_m a_m e^{-i\vec{J}\cdot\hat{n}\theta}|jm\rangle.$$

$e^{-i\vec{J}\cdot\hat{n}\theta} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}(\vec{J}\cdot\hat{n})^n \Rightarrow$ a function of $J_z, J_\pm$, and we know $J_z, J_\pm$ do not change the value of j. Thus $e^{-i\vec{J}\cdot\hat{n}\theta}|jm\rangle$ remains inside L^j, and L^j is invariant under rotation.

The proof of the irreducibility is more tricky. It means for any state in L^j, by acting $J_\pm$ successively, we can arrive at $2j+1$ linearly independent states. Thus there's no a smaller invariant subspace inside L^j. We will not give a rigours proof here.

§ Representation of rotation group

Any rotation $g(\hat{n},\theta)$ can be represented as a 3×3 orthogonal matrix, and mathmatily called $SO(3)$ group, or, isomorphically, $SU(2)$. The only difference between $SO(3)$ and $SU(2)$ is that $SU(2)$ includes both half integer and integer angular momentum, and SO(3) only includes integer.

Loosely speaking, we often do not distinguish this difference. Check Sakurai Sect 3.3 for more info.

Quantum mechanically, g is represented by the rotation operater $D(g) = e^{-i\vec{J}\cdot\hat{n}\theta}$. In the space of L^j defined above, $D(g)$ is further represented by a $(2j+1)\times(2j+1)$ matrix as

$$\boxed{D^j_{m'm}(g) = \langle jm' | e^{-i\vec{J}\cdot\hat{n}\theta} | jm\rangle}$$

The correspondence between $g \to D(g) \to D^j_{m'm}(g)$ follows the product of matrix:

$$\begin{array}{ccccc} g & \longrightarrow & D(g) & \longrightarrow & D^j_{m'm}(g) \\ g = g_1 g_2 & \longrightarrow & D(g) = D(g_1)D(g_2) & \longrightarrow & D^j_{m'm}(g) = \sum_{m''} D^j_{m'm''}(g_1) D^j_{m''m}(g_2) \end{array}$$

rotation operation $\longrightarrow$ QM rotation operator $\longrightarrow$ Rotation D-matrix in the space L^j.

Ex1: please prove that $D^j_{m'm}(g)$ is a unitary matrix.

§ Calculation of $D^j_{m'm}(g)$.

The parameterization of $g(\hat{n}, \theta)$ is not convenient for later use. We use the Eulerian angles, which connects body frame and lab frame nicely. Now we define $g(\alpha\,\beta\,\gamma)$, as three-step rotation:

① rotate around z-axis at α-angle, then $(\hat{x}\,\hat{y}\,\hat{z}) \xrightarrow{g(\hat{z},\alpha)} (\hat{x}', \hat{y}', \hat{z}'=\hat{z})$

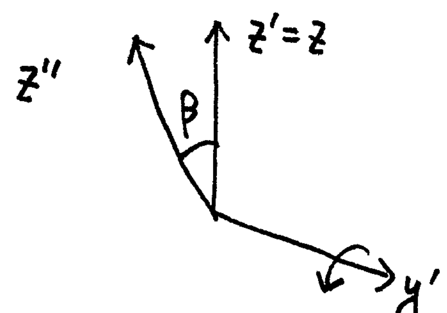
② rotate around y'-axis with β-angle, then $(\hat{x}', \hat{y}', \hat{z}') \xrightarrow{g(y',\beta)} (\hat{x}'', \hat{y}''=\hat{y}', \hat{z}''$
(NOT y-axis), but the new y-axis)

③ rotate around z''-axis, then $(\hat{x}'', \hat{y}'', \hat{z}'') \xrightarrow{g(\hat{z}'',\gamma)} (\hat{x}''', \hat{y}''', \hat{z}'''=\hat{z}'')$
(the new z-axis)

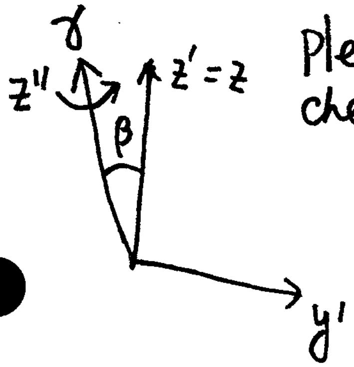
Thus $g(\alpha\,\beta\,\gamma) = g(\hat{z}'', \gamma)\, g(\hat{y}', \beta)\, g(\hat{z}, \alpha)$

Let's check the relation $g(\hat{z}'', \gamma)$ and $g(\hat{z}, \gamma)$.

rotation $g(\hat{y}', \beta)$ apply on $z \longrightarrow z''$, i.e. $\hat{z}'' = g(\hat{y}', \beta)\, \hat{z}$.



From the rotation theory. we have $\boxed{g(\hat{z}'', \gamma) = g(y'\beta)\, g(z,\gamma)\, g^{-1}(y', \beta)}$.



Please check! =

step 1 apply $g^{-1}(y', \beta)$ such that $\hat{z}''$-axis is restored to $\hat{z}$.

↓

step 2 apply rotation around $\hat{z}$ with same angle γ

↓

step 3 rotate the system back by $g(y', \beta)$

$$\Rightarrow \quad g(\hat{z}'', \gamma)\, g(y', \beta) = g(y'\beta)\, g(z\, \gamma)$$

$$\Rightarrow \quad g(\alpha \beta \gamma) = g(\hat{y}'\beta)\, g(\hat{z}, \alpha)\, g(\hat{z}, \gamma)$$

Ex2: please prove $\boxed{g(\hat{y}'\beta)\, g(\hat{z}, \alpha) = g(\hat{z}, \alpha)\, g(\hat{y}, \beta)}$

Finally, we have $\boxed{g(\alpha \beta \gamma) = g(\hat{z}, \alpha)\, g(\hat{y}, \beta)\, g(\hat{z}, \gamma)}$.

The 3×3 matrix Rep for g is

$$g(\hat{z}, \theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad g(\hat{y}, \theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Now we map to the D-matrix

$$D(g) = D(g(\hat{z}\, \alpha))\, D(g(\hat{y}, \beta))\, D(g(\hat{z}, \gamma)) = e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}$$

$$\Rightarrow D^j_{m'm}(\alpha\beta\gamma) = \langle jm' |\, e^{-iJ_z\alpha}\, e^{-iJ_y\beta}\, e^{-iJ_z\gamma}\, | jm \rangle$$

$$= e^{-im'\alpha - im\gamma} \langle jm' |\, e^{-iJ_y\beta}\, | jm \rangle$$

we define d-matrix: $\boxed{d^j_{m'm}(\beta) = \langle jm' |\, e^{-iJ_y\beta}\, | jm \rangle}$

iJ_y in the representation of $|jm\rangle$, i.e $i\langle jm' | J_y | jm \rangle$ are purely real, so does $\langle jm' |\, e^{-iJ_y\beta}\, | jm \rangle$. $\Rightarrow \left(d^j_{m'm}(\beta)\right)^* = d^j_{m'm}(\beta)$

Ex3: prove that $\boxed{d^j_{mm'}(-\beta) = d^j_{m'm}(\beta)}$.

§ Expression of $d^j_{m'm}(\beta)$.

we use a 2D harmonic oscillator to represent algebra of angular momentum.

The creation/anihilation operators $a_1\ a_1^\dagger$, $a_2\ a_2^\dagger$ can represent J_x, J_y, J_z

follow $\vec{J} = \frac{1}{2}(a_1^\dagger a_2^\dagger)\,\vec{\sigma}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ or

$$J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$$

$$J_x = \frac{1}{2}(a_1^\dagger a_2 + a_2^\dagger a_1)$$

$$J_y = \frac{-i}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)$$

$\boxed{\text{Schwinger boson Rep.}}$

Ex4: please check the expressions of $\vec{J}$ in terms of $a_{1,2}, a_{1,2}^\dagger$ satisfy $[J_i, J_j] = i\epsilon_{ijk}J_k$, and also $\vec{J}^2 = \left(\frac{a_1^\dagger a_1 + a_2^\dagger a_2}{2}\right)\left(\frac{a_1^\dagger a_1 + a_2^\dagger a_2}{2} + 1\right)$.

Next is the map of states. $|jm\rangle$ corresponds to the state with $n_1 = a_1^\dagger a_1 = j+m$,

$n_2 = a_2^\dagger a_2 = j-m$, i.e

$$|jm\rangle = \frac{(a_1^\dagger)^{j+m}(a_2^\dagger)^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle = |n_1 n_2\rangle.$$

Ex5: Check that in the Schwinger boson Rep. We do have

$$\begin{cases} J_z|jm\rangle = m|jm\rangle \\ J_\pm|jm\rangle = \sqrt{(j\mp m)(j\mp m+1)}\,|jm\pm1\rangle \end{cases}$$

← Prove them in the Schwinger boson Rep.

$$e^{-iJ_y\beta}|jm\rangle = \frac{e^{-iJ_y\beta}(a_1^+)^{j+m}(a_2^+)^{j-m}\,e^{iJ_y\beta}\;e^{-iJ_y\beta}|0\rangle}{\sqrt{(j+m)!\,(j-m)!}}$$

define $\begin{pmatrix} a_1^{+\prime} \\ a_2^{+\prime} \end{pmatrix} = e^{-iJ_y\beta}\begin{pmatrix} a_1^+ \\ a_2^+ \end{pmatrix} e^{iJ_y\beta}$, we have

$$e^{-iJ_y\beta}|jm\rangle = \frac{(a_1^{+\prime})^{j+m}(a_2^{+\prime})^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle$$

check the note before. $a_1^{+\prime} = a_1^+\cos\frac{\beta}{2} + a_2^+\sin\frac{\beta}{2}$

$a_2^{\prime +} = -a_1^+\sin\frac{\beta}{2} + a_2^+\cos\frac{\beta}{2}$

$$\Rightarrow\; e^{-iJ_y\beta}|jm\rangle = \frac{(a_1^+\cos\frac{\beta}{2} + a_2^+\sin\frac{\beta}{2})^{j+m}(-a_1^+\sin\frac{\beta}{2} + a_2^+\cos\frac{\beta}{2})^{j-m}}{\sqrt{(j+m)!\,(j-m)!}}|0\rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}}\sum_{m'=-j}^{j}\sum_{\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}\left(a_1^+\cos\frac{\beta}{2}\right)^{m+m'+\sigma}\left(a_2^+\sin\frac{\beta}{2}\right)^{j-m'-\sigma}$$
$$\otimes\,(-)^{j-m-\sigma}\left(a_1^+\sin\frac{\beta}{2}\right)^{j-m-\sigma}\left(a_2^+\cos\frac{\beta}{2}\right)^{\sigma}|0\rangle$$

$$= \frac{1}{\sqrt{(j+m)!\,(j-m)!}}\sum_{m'=-j}^{j}\sum_{\sigma}\binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma}(a_1^+)^{j+m'}(a_2^+)^{j-m'}$$
$$(-)^{j-m-\sigma}\left(\cos\frac{\beta}{2}\right)^{m+m'+2\sigma}\left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m'-m}|0\rangle$$

$$\left.\begin{array}{l} 0 \le \sigma \le j-m \\ -m-m' \le \sigma \le j-m' \end{array}\right\} \Rightarrow \max(0, -(m+m')) \le \sigma \le \min(j-m, j-m')$$

$$\Rightarrow \boxed{d^j_{m'm}(\beta) = \sqrt{\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}} \sum_\sigma \binom{j+m}{j-m'-\sigma}\binom{j-m}{\sigma} (-)^{j-m-\sigma} \left(\cos\frac{\beta}{2}\right)^{m+m'+2\sigma} \left(\sin\frac{\beta}{2}\right)^{2j-2\sigma-m-m'}}$$

§: Important relations without proof. (l is integer below)

$$d^l_{0m}(\beta) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P^m_l(\cos\beta)$$

$$D^l_{00}(\alpha\beta\gamma) = d^l_{00}(\beta) = P_l(\cos\beta)$$

we will prove $D^l_{m0}(\alpha\ \beta.\ \gamma=0) = \sqrt{\frac{4\pi}{2l+1}}\ Y^*_{lm}(\theta=\beta.\ \phi=\alpha)$

Define $|\hat{n}\rangle$ as direction eigenket, where $\hat{n}(\theta.\varphi)$ along any solid angle direction. For ~~state~~ $|lm\rangle$, we have $\boxed{\langle\hat{n}|lm\rangle = Y_{lm}(\theta.\varphi) = Y_{lm}(\hat{n})}$

$$|n\rangle = e^{-iJ_z\phi}\ e^{-iJ_y\theta}\ |\hat{z}\rangle$$

$$= \sum_{l'}\sum_{m'} D[g(\alpha=\phi,\ \beta=\theta.\ \gamma=0)]\ |l'm'\rangle\langle l'm'|\hat{z}\rangle$$

$$\langle lm'|n\rangle = \sum_{l'm} \langle lm'|D(g)|l'm\rangle\langle l'm|\hat{z}\rangle$$

$$= \sum_m \langle lm'|D(g)|lm\rangle\langle lm|\hat{z}\rangle = \sum_m D^l_{m'm}(\alpha=\phi.\ \beta=\theta.\ \gamma=0)\ \langle lm|\hat{z}\rangle$$

$$\Rightarrow Y^*_{lm'}(\theta.\varphi) = \sum_m D^l_{m'm}(\alpha=\phi.\ \beta=\theta.\ \gamma=0)\ Y^*_{lm}(\theta=0,\ \varphi \text{ undetermined})$$

$$Y_{lm}(\theta=0,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\ P_l(\cos\theta)\Big|_{\cos\theta=1} \delta_{m,0} = \sqrt{\frac{2l+1}{4\pi}}\ \delta_{m0}$$

$$\Rightarrow Y^*_{\ell m'}(\theta.\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, D^{\ell}_{m',0}(\alpha=\phi,\ \beta=\theta,\ \gamma=0)$$

or
$$\boxed{D^{\ell}_{m0}(\alpha\,\beta\,\gamma=0) = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m}(\theta.\varphi)\Big|_{\theta=\beta,\ \phi=\alpha}}$$

Set $m=0$ $\Rightarrow$ and use $Y_{\ell 0}(\theta.\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$ $\Rightarrow$

$$\boxed{d^{\ell}_{00}(\beta) = P_\ell(\cos\theta)\Big|_{\theta=\beta}}$$