Week 4 Discussion Session

2014.1.25

Homework #2

Problem 1. (Sakurai 4.4)

Solution:

(a) This is a CG coefficient problem. We are asked to do a basis transformation from $|(l_1, m_1); (l_2 = \frac{1}{2}, m_2)\rangle$ to $|l_1 l_2; jm\rangle$.

By checking the CG coefficient table, one gets

$$\mathcal{Y}_{l=0}^{j=1/2,\, m=1/2}(\vec{r}) = \sqrt{\frac{1}{4\pi}}\,|\uparrow\rangle \cancel{(= Y_{00} \otimes |\uparrow\rangle)} \quad (= \langle \vec{r}\,|(Y_{00} \otimes |\uparrow\rangle)\ ).$$

(b) Both $\vec{\sigma}$ and $\vec{r}$ are defined in terms of the original basis. So we first expand $\mathcal{Y}_{l=0}^{j=1/2,m=1/2}$ in the old basis, then perform the action of $\vec{\sigma}\cdot\vec{r}$, and finally transform back into the new basis via CG coefficients.

$$\begin{aligned}
&\langle \vec{r}\,|\,\vec{\sigma}\cdot\vec{r}\,|\mathcal{Y}_{l=0}^{j=1/2,m=1/2}\rangle \\
&= \vec{\sigma}\cdot\vec{r}\ \mathcal{Y}_{l=0}^{j=1/2,m=1/2}(\vec{r}) \\
&= \vec{\sigma}\cdot\vec{r}\ \sqrt{\frac{1}{4\pi}}\ |\uparrow\rangle \\
&= \sqrt{\frac{1}{4\pi}}\,(x\sigma_x + y\sigma_y + z\sigma_z)\ |\uparrow\rangle \\
&= \sqrt{\frac{1}{4\pi}}\,(x|\downarrow\rangle + iy|\downarrow\rangle + z|\uparrow\rangle) \\
&= \sqrt{\frac{1}{4\pi}}\,r\,(\sin\theta(\cos\phi + i\sin\phi)\,|\downarrow\rangle + \cos\theta\,|\uparrow\rangle) \\
&= \sqrt{\frac{1}{4\pi}}\,r\,(\cos\theta\,|\uparrow\rangle + \sin\theta\, e^{i\phi}\,|\downarrow\rangle)
\end{aligned}$$

By checking the spherical harmonics table, we find

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

Hence the above expression

$$\begin{aligned}
&= \sqrt{\frac{1}{4\pi}}\,r\left(\sqrt{\frac{4\pi}{3}}\,Y_{10}|\uparrow\rangle - \sqrt{\frac{8\pi}{3}}\,Y_{11}\ |\downarrow\rangle\right) \\
&= r\left(\frac{1}{\sqrt{3}}Y_{10}|\uparrow\rangle - \sqrt{\frac{2}{3}}\,Y_{11}|\downarrow\rangle\right)
\end{aligned}$$

By checking the CG coefficients,

$$\mathcal{Y}_{l}^{j=l\pm 1/2,\,m} = \pm\sqrt{\frac{l\pm m+\frac{1}{2}}{2l+1}}\ Y_{l,m-1/2}\ |\uparrow\rangle + \sqrt{\frac{l\mp m+\frac{1}{2}}{2l+1}}\ Y_{l,m+1/2}\ |\downarrow\rangle$$

Plugging in $l=1$, $m=\frac{1}{2}$, $j=1-\frac{1}{2}=\frac{1}{2}$, one gets

$$\mathcal{Y}_{l=1}^{j=1/2,\,m=1/2} = -\frac{1}{\sqrt{3}}Y_{10}|\uparrow\rangle + \sqrt{\frac{2}{3}}\,Y_{11}|\downarrow\rangle.$$

Thus $\vec{\sigma}\cdot\vec{r}\ \mathcal{Y}_{l=0}^{j=1/2,m=1/2}(\vec{r}) = -r\ \mathcal{Y}_{l=1}^{j=1/2,m=1/2}(\vec{r})$.

$\sqcap$

(c) Denote the angular momentum operators in real space to be $\vec{L}\ (=\hat{\vec{r}}\times\hat{\vec{p}})$, in spin space to be $\vec{S}$. Then a rotation in real space is $e^{-i\vec{L}\cdot\hat{n}\theta}$, in spin space is $e^{-i\vec{S}\cdot\hat{n}\theta}$. The total rotation is $e^{-i\vec{L}\cdot\hat{n}\theta}\otimes e^{-i\vec{S}\cdot\hat{n}\theta} = e^{-i\vec{L}\cdot\hat{n}\theta}e^{-i\vec{S}\cdot\hat{n}\theta}$ (We have identified $e^{-i\vec{L}\cdot\hat{n}\theta}$ with $e^{-i\vec{L}\cdot\hat{n}\theta}\otimes\mathbb{1}$, and $e^{-i\vec{S}\cdot\hat{n}\theta}$ with $\mathbb{1}\otimes e^{-i\vec{S}\cdot\hat{n}\theta}$).

We claim that $\vec{\sigma}\cdot\hat{\vec{r}}$ is invariant under total rotations. hence it is a rank-0 irreducible operator under total rotations.

Notice that $\vec{\sigma}$ is a set of rank-1 irreducible tensor operators under spin rotations, i.e. $e^{-i\vec{S}\cdot\hat{n}\theta}\,\vec{\sigma}\,e^{i\vec{S}\cdot\hat{n}\theta} = R^{-1}\vec{\sigma}$ ($R=R(\hat{n},\theta)$ is the rotation in $\mathbb{R}^3$ around direction $\hat{n}$ by an angle θ), while $\hat{\vec{r}}$ is a set of rank-1 irreducible tensor operators under space rotations, i.e. $e^{-i\vec{L}\cdot\hat{n}\theta}\,\hat{\vec{r}}\,e^{i\vec{L}\cdot\hat{n}\theta} = R^{-1}\hat{\vec{r}}$.

Thus
$$
\begin{aligned}
&(e^{-i\vec{L}\cdot\hat{n}\theta}e^{-i\vec{S}\cdot\hat{n}\theta})(\vec{\sigma}\cdot\hat{\vec{r}})(e^{-i\vec{L}\cdot\hat{n}\theta}e^{-i\vec{S}\cdot\hat{n}\theta})^\dagger \\
&= (e^{-i\vec{L}\cdot\hat{n}\theta}\,\vec{\sigma}\,e^{i\vec{L}\cdot\hat{n}\theta})\cdot(e^{-i\vec{S}\cdot\hat{n}\theta}\,\hat{\vec{r}}\,e^{i\vec{S}\cdot\hat{n}\theta}) \\
&= R^{-1}\vec{\sigma}\cdot R^{-1}\hat{\vec{r}} \\
&= \vec{\sigma}\cdot\hat{\vec{r}},
\end{aligned}
$$
completing the proof of the claim.

By Wigner-Eckart theorem, it is easy to see that $\vec{\sigma}\cdot\hat{\vec{r}}\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}$ must have quantum numbers $j'=j=1/2$, $m'=m=1/2$. While ℓ is still undetermined

I.e. $\vec{\sigma}\cdot\hat{\vec{r}}\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0} = \sum_\ell C_\ell\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell}$
$$= C_0\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0} + C_1\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=1}$$

Now consider parity transformation. First notice that $\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}$ has even parity (i.e. is an eigenvector of parity transformation P with eigenvalue $+1$) under the convention that P is an identity transformation on spin-1/2 space. This is because $\langle\vec{r}|P|\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}\rangle = \langle -\vec{r}|\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}\rangle = \mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}(-\vec{r})$
$$= \mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}(\vec{r}).$$

We claim that $\vec{\sigma}\cdot\hat{\vec{r}}\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}$ is of odd parity.

Check:
$$
\begin{aligned}
P\vec{\sigma}\cdot\hat{\vec{r}}\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0} &= P(\vec{\sigma}\cdot\hat{\vec{r}})P^{-1}\cdot P\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0} \\
&= P\vec{\sigma}P^{-1}\cdot P\hat{\vec{r}}P^{-1}\cdot P\,\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0} \\
&= \vec{\sigma}\cdot(-\hat{\vec{r}})\cdot\mathcal{Y}^{j=1/2,m=1/2}_{\ell=0}
\end{aligned}
$$

$$= - \vec{\sigma}\cdot\hat{\vec{r}}\ \mathcal{Y}^{j=1/2,\, m=1/2}_{l=0}$$

This shows that in the expansion $\vec{\sigma}\cdot\hat{\vec{r}}\ \mathcal{Y}^{j=1/2,\, m=1/2}_{l=0} = C_0\, \mathcal{Y}^{j=1/2,\, m=1/2}_{l=0} + C_1\, \mathcal{Y}^{j=1/2,\, m=1/2}_{l=1}$,
only the second term survives, i.e.

$$\vec{\sigma}\cdot\hat{\vec{r}}\ \mathcal{Y}^{j=1/2,\, m=1/2}_{l=0} = C_1\, \mathcal{Y}^{j=1/2,\, m=1/2}_{l=1} .$$

By the calculation in part (b), $C_1 = -1$.

Problem 2. (Sakurai 4.10)

Solution:

(c) We first do part (c) and then move onto parts (a) and (b).

Step 1. T sends $|jm\rangle$ to $|j,-m\rangle$, i.e. $T|jm\rangle = C_m|j,-m\rangle$, $|C_m|^2 = 1$.

Proof.
$$\begin{aligned} J_z T|jm\rangle &= -TJ_z|jm\rangle \quad (\text{by using } T\vec{J}T^{-1} = -\vec{J}) \\ &= -T(m|jm\rangle) \\ &= -m\, T|jm\rangle \end{aligned}$$
$$\Rightarrow T|jm\rangle = C_m|j,-m\rangle$$

$|C_m|^2 = 1$ is because T is anti-unitary, i.e.
$$\begin{aligned} \langle jm|jm\rangle &= \langle jm|T^{\dagger}T|jm\rangle \\ &= \langle Tjm|Tjm\rangle \\ &= |C_m|^2\langle j,-m|j,-m\rangle \end{aligned}$$
$$\Rightarrow |C_m|^2 = 1.$$

Step 2. $C_m = -C_{m-1}$

Proof. $J_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\;|j,m-1\rangle$
$$\Rightarrow TJ_-|jm\rangle = \sqrt{(j+m)(j-m+1)}\;T|j,m-1\rangle$$
$$\Rightarrow T(J_x - iJ_y)|jm\rangle = \sqrt{(j+m)(j-m+1)}\;C_{m-1}|j,-m+1\rangle$$
$$\Rightarrow (-J_x - (-i)(-J_y))\,T|jm\rangle = \sqrt{(j+m)(j-m+1)}\;C_{m-1}|j,-m+1\rangle$$
$$\Rightarrow -(J_x + iJ_y)\,C_m|j,-m\rangle = \sqrt{(j+m)(j-m+1)}\;C_{m-1}|j,-m+1\rangle$$
$$\Rightarrow -C_m\sqrt{(j-(-m))(j+(-m)+1)} = C_{m-1}\sqrt{(j+m)(j-m+1)}\;|j,-m+1\rangle$$
$$\Rightarrow -C_m\sqrt{(j+m)(j-m+1)}\;|j,-m+1\rangle = C_{m-1}\sqrt{(j+m)(j-m+1)}\;|j,-m+1\rangle$$
$$\Rightarrow C_m = -C_{m-1}$$

The overall phase is not important. A usual convention ~~would~~ is $C_j = 1$. Here $T|j,m\rangle = i^{2m}|j,-m\rangle$ ~~$= (-1)^m|j,-m\rangle$~~ is some other convention, but the point is that $|C_m|^2 = 1$ and $C_m = -C_{m-1}$ are always satisfied.

(a)
$$\begin{aligned} &T\mathcal{D}(R)|j,m\rangle \\ &= Te^{-i\vec{J}\cdot\hat{n}\theta}|j,m\rangle \\ &= Te^{-i\vec{J}\cdot\hat{n}\theta}T^{-1}\cdot T|j,m\rangle \\ &= e^{-i\vec{J}\cdot\hat{n}\theta}\,i^{2m}|j,-m\rangle \\ &= i^{2m}\,\mathcal{D}(R)|j,-m\rangle \end{aligned}$$

(b) $\mathcal{D}^{(j)*}_{m'm}(R) = \left(\langle m'| e^{-i\vec{J}\cdot\hat{n}\theta} |m\rangle\right)^*$

$= \left(\langle m'| T^{-1} \cdot T e^{-i\vec{J}\cdot\hat{n}\theta} T^{-1} \cdot T |m\rangle\right)^*$

$= \left(\langle m'| T^{\dagger} \cdot e^{-i\vec{J}\cdot\hat{n}\theta} \cdot T|m\rangle\right)^*$

$= \langle Tm'| e^{-i\vec{J}\cdot\hat{n}\theta} T|m\rangle$

$= (i^{2m'})^* \langle -m'| e^{-i\vec{J}\cdot\hat{n}\theta} | i^{2m} |-m\rangle$

$= i^{2(m-m')} \mathcal{D}^{(j)}_{-m',-m}(R)$

Notice that $m-m'$ is always an integer, hence there is no ambiguity writing in the following form

$= (-1)^{m-m'} \mathcal{D}^{(j)}_{-m',-m}(R)$

Problem 3. Sakurai 4.11

Solution:

Since $THT^{-1} = H$, i.e. $HT = TH$. we have for any eigenstate $|\Psi_E\rangle$ of H with energy E, $T|\Psi_E\rangle$ is still an eigenvector of energy E. But we have assumed that all energy levels are non-degenerate, hence $T|\Psi_E\rangle$ can only differ by a phase, i.e. $T|\Psi_E\rangle = e^{i\delta_E}|\Psi_E\rangle$.

Then $\langle \Psi_E | T^{-1}\vec{L}\, T|\Psi_E\rangle$

$$= \langle \Psi_E | T^{+}\vec{L}\, T|\Psi_E\rangle \begin{cases} = \langle \Psi_E | (-\vec{L}) | \Psi_E\rangle = -\langle \Psi_E | \vec{L} | \Psi_E\rangle \\ = (\langle T\Psi_E | \vec{L} | T\Psi_E\rangle)^* = \langle \Psi_E | \vec{L} | \Psi_E\rangle \end{cases}$$

$$\Rightarrow \langle \Psi_E | \vec{L} | \Psi_E\rangle = -\langle \Psi_E | \vec{L} | \Psi_E\rangle = 0.$$

Now we know $T|\Psi_E\rangle = e^{i\delta_E}|\Psi_E\rangle$, then

$$\cancel{T\sum_{\ell,m} F_{\ell m}(r)\, Y_{\ell m}(\theta,\phi)}$$

$$\cancel{= \sum_{\ell,m} F^*_{\ell m}(r)\, T\, Y_{\ell m}(\theta,\phi)}$$

$$T\sum_{\ell}\sum_{m} F_{\ell m} Y_{\ell m} = e^{i\delta}\sum_{\ell}\sum_{m} F_{\ell m} Y_{\ell m}$$

$$\Rightarrow \langle \vec{r}|T|\sum_{\ell,m} F_{\ell m} Y_{\ell m}\rangle = e^{i\delta}\sum_{\ell}\sum_{m} F_{\ell m} Y_{\ell m}$$

$$LHS = (\langle T\vec{r}\,|\sum_{\ell,m} F_{\ell m} Y_{\ell m}\rangle)^*$$

$$= (\langle \vec{r}\,|\sum_{\ell,m} F_{\ell m} Y_{\ell m}\rangle)^*$$

$$= \sum_{\ell,m} F^*_{\ell m}(r)\; Y^*_{\ell m}(\theta,\phi)$$

$$= \sum_{\ell,m} F^*_{\ell m}(r)\; (-1)^m\, Y_{\ell,-m}(\theta,\phi)$$

$$= \sum_{\ell,m'} F^*_{\ell,-m'}(r)\; (-1)^{-m'}\, Y_{\ell,m'}(\theta,\phi)$$

$$= \sum_{\ell,m} F^*_{\ell,-m}(r)\; (-1)^{m}\, Y_{\ell,m}(\theta,\phi)$$

$$\Rightarrow e^{i\delta} F_{\ell m} = (-1)^m F^*_{\ell,-m}(r)$$