Homework 1. 2014.1.11

Problem 1.

Solution:

a. $\sum_{m=-j}^{+j} |d^{(j)}_{mm'}(\beta)|^2 \cdot m$

$= \sum_{m=-j}^{+j} d^{(j)*}_{mm'}(\beta)\, d^{(j)}_{mm'}(\beta) \cdot m$

$= \sum_{m=-j}^{+j} d^{(j)\dagger}_{m'm}(\beta)\, d^{(j)}_{mm'}(\beta) \cdot m$

$= \sum_{m=-j}^{+j} d^{(j)}_{m'm}(-\beta)\, d^{(j)}_{mm'}(\beta) \cdot m$

$= \sum_{m=-j}^{+j} \langle m' | e^{iJ_y\beta} | m\rangle\, m\, \langle m | e^{-iJ_y\beta} | m'\rangle$

$= \langle m' | e^{iJ_y\beta} J_z e^{-iJ_y\beta} | m'\rangle$

$= \langle m' | J_z \cos\beta + J_x \sin\beta | m'\rangle$ (since $\{J_x, J_y, J_z\}$ constitute of a set of rank-1 irreducible tensor operators)

$= m'\cos\beta + \langle m' | J_x | m'\rangle \sin\beta$

But $\langle m' | J_x | m'\rangle = \langle m' | \frac{1}{2}(J_+ + J_-) | m'\rangle = 0.$

hence

$$\sum_{m=-j}^{+j} |d^{(j)}_{mm'}(\beta)|^2 \cdot m = m'\cos\beta$$

b. Similarly, $\sum_{m=-j}^{+j} m^2 |d^{(j)}_{m'm}(\beta)|^2 = \langle m' | e^{iJ_y\beta} J_z^2 e^{-iJ_y\beta} | m'\rangle$.

Now

$\langle m' | e^{iJ_y\beta} J_z^2 e^{-iJ_y\beta} | m'\rangle$

$= \langle m' | e^{iJ_y\beta} J_z e^{-iJ_y\beta} \cdot e^{iJ_y\beta} J_z e^{-iJ_y\beta} | m'\rangle$

$= \langle m' | (J_z\cos\beta + J_x\sin\beta)(J_z\cos\beta + J_x\sin\beta) | m'\rangle$

in which

$(J_z\cos\beta + J_x\sin\beta)|m'\rangle = \left(m'\cos\beta + \frac{1}{2}\sin\beta\,(J_+ + J_-)\right)|m'\rangle$

$= m'\cos\beta|m'\rangle + \frac{1}{2}\sin\beta\sqrt{(j-m')(j+m'+1)}\,|m'+1\rangle + \frac{1}{2}\sin\beta\sqrt{(j+m')(j-m'+1)}\,|m'-1\rangle$

Hence

$\langle m' | (J_z\cos\beta + J_x\sin\beta)(J_z\cos\beta + J_x\sin\beta) | m'\rangle$

$= m'^2\cos^2\beta + \frac{1}{4}\sin^2\beta\left[(j-m')(j+m'+1) + (j+m')(j-m'+1)\right]$

$= m'^2\cos^2\beta + \frac{1}{4}\sin^2\beta\left[2(j^2-m'^2) + 2j\right]$

$= \frac{1}{2}j(j+1)\sin^2\beta + m'^2\frac{1}{2}(3\cos^2\beta - 1)$

Problem 2 of Homework # 1. (Sakurai 3.32)

(a) This is just a basis transformation.

One way to solve it.

Check the spherical harmonics table

$$r^2 Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}}\,(x \pm iy)^2$$

$$r^2 Y_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\, z(x \pm iy)$$

$$r^2 Y_{2,0} = \sqrt{\frac{5}{16\pi}}\,(3z^2 - r^2) = \sqrt{\frac{5}{16\pi}}\,(2z^2 - x^2 - y^2)$$

$$r^2 Y_{00} = \sqrt{\frac{1}{4\pi}}\, r^2 = \sqrt{\frac{1}{4\pi}}\,(x^2+y^2+z^2) .$$

This is just a transformation from the basis $\{x^2, y^2, z^2, xy, xz, yz\}$ to $\{Y_{00}, Y_{2,2}, Y_{2,1}, Y_{2,0}, Y_{2,-1}, Y_{2,-2}\}$.

Now xy, xz, x^2-y^2 is expressed in terms of the 1st set of basis, invert the transformation matrix one can get the expression for them in the ~~basis~~ new basis.

More specifically, suppose the transformation matrix is M, i.e.

$$(Y_{00}\ \ Y_{2,2}\ \ Y_{2,1}\ \ Y_{2,0}\ \ Y_{2,-1}\ \ Y_{2,-2}) = (x^2\ \ y^2\ \ z^2\ \ xy\ \ xz\ \ yz)\, M .$$

Then

$$xy = (x^2\ \ y^2\ \ z^2\ \ xy\ \ xz\ \ yz)\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix}$$

$$\cancel{= aY_{00}\ Y_{2,2}\ Y_{2,0}\ Y_{2}}$$

$$= (Y_{00}\ \ Y_{2,2}\ \ Y_{2,1}\ \ Y_{2,0}\ \ Y_{2,-1}\ \ Y_{2,-2})\, M^{-1}\begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} .$$

The results are

$$xy = i\sqrt{\frac{2\pi}{15}}\, r^2(Y_{2,-2} - Y_{2,2})$$

$$xz = \sqrt{\frac{2\pi}{15}}\, r^2(Y_{2,-1} - Y_{2,+1})$$

$$x^2 - y^2 = \sqrt{\frac{8\pi}{15}}\, r^2(Y_{2,2} + Y_{2,-2})$$

The point is that all of these three 2nd order homogeneous polynomials have no component on Y_{00}, although an arbitrary expression might have.

(b) First we introduce the following notation for CG coefficients

$$\langle (j,m) \,|\, (j_1,m_1)\,;\,(j_2,m_2) \rangle$$

Then

$$\begin{aligned}
Q &= e\,\langle \alpha,\,(j,j)\,|\;3z^2-r^2\;|\,\alpha,\,(j,j)\rangle \\
&= e\,\langle \alpha,\,(j,j)|\;\sqrt{\tfrac{16\pi}{5}}\;r^2\,\big(Y_{20}\,|\alpha,\,(j,j)\rangle\big) \\
&= e\,\langle \alpha,(j,j)|\;\sqrt{\tfrac{16\pi}{5}}\;r^2 \sum_{|j-2|\leq j'\leq |j+2|} |(Y\alpha),\,(j',j)\rangle\langle (j',j)\,|\,(2,0);\,(j,j)\rangle \\
&= e\sqrt{\tfrac{16\pi}{5}}\;\langle \alpha,(j,j)|\,r^2\,|\,(Y\alpha),\;(j,j)\rangle\;\langle (j,j)\,|\,(2,0);\,(j,j)\rangle
\end{aligned}$$

in which

$$\langle \alpha,(j,j)|\,r^2\,|\,(Y\alpha),\,(j,j)\rangle\;\big(=\langle \alpha,(j,m)\,|\,r^2\,|\,(Y\alpha),\,(j,m)\rangle\big)$$

$$= f(\alpha,j)$$

(this is identical with the conclusion in lectures by defining $r^2|\alpha,(j,j)\rangle = |\alpha',\,(j,j)\rangle$).

$\Rightarrow$

$$Q = e\sqrt{\tfrac{16\pi}{5}}\;f(\alpha,j)\;\langle (j,j)\,|\,(2,0)\,;\,(j,j)\rangle$$

$$\begin{aligned}
&e\,\langle \alpha,(j,m')|\,(x^2-y^2)\,|\,\alpha,(j,j)\rangle \\
&= e\,\langle \alpha,\,(j,m')|\;\sqrt{\tfrac{8\pi}{15}}\;r^2\,(Y_{2,2}+Y_{2,-2})\;|\,\alpha,(j,j)\rangle \\
&= e\sqrt{\tfrac{8\pi}{15}}\;\langle \alpha,(j,m')|\,r^2 \sum_{|j-2|\leq j'\leq |j+2|}\Big(|(Y\alpha),(j',\,j+2)\rangle\langle (j',\,j+2)\,|\,(2,2);\,(j,j)\rangle \\
&\qquad\qquad + |(Y\alpha),(j',\;j-2)\rangle\langle (j',\,j-2)\,|\,(2,-2);\,(j,j)\rangle\Big) \\
&= e\sqrt{\tfrac{8\pi}{15}}\;\langle \alpha,\,(j,\,j-2)|\,r^2\,|(Y\alpha),(j,\;j-2)\rangle\;\langle (j,\,j-2)\,|\,(2,-2);\,(j,j)\rangle\;\delta_{m',j-2} \\
&= e\sqrt{\tfrac{8\pi}{15}}\;f(\alpha,j)\;\delta_{m',j-2}\;\langle (j,j-2)\,|\,(2,-2)\,;\,(j,j)\rangle
\end{aligned}$$

Thus

$$\begin{aligned}
&e\,\langle \alpha,(j,m')\,|\,(x^2-y^2)\,|\,\alpha,\,(j,j)\rangle \\
&= \frac{1}{\sqrt{6}}\;\delta_{m',j-2}\;\frac{\langle (j,j-2)\,|\,(2,-2)\,;\,(j,j)\rangle}{\langle (j,j)\,|\,(2,0)\,;\,(j,j)\rangle}\;Q
\end{aligned}$$

Problem 3. (Sakurai 3.33)

Solution: For simplicity, denote

$$\alpha_1 = \frac{eQ}{2s(s-1)\hbar^2}\left(\frac{\partial^2\phi}{\partial x^2}\right)_0 , \quad \alpha_2 = \frac{eQ}{2s(s-1)\hbar^2}\left(\frac{\partial^2\phi}{\partial y^2}\right)_0 , \quad \alpha_3 = \frac{eQ}{2s(s-1)\hbar^2}\left(\frac{\partial^2\phi}{\partial z^2}\right)_0$$

Then

$$\begin{aligned} H_{int} &= \alpha_1 S_x^2 + \alpha_2 S_y^2 + \alpha_3 S_z^2 \\ &= \alpha_1\left(\tfrac{1}{2}(S_+ + S_-)\right)^2 + \alpha_2\left(\tfrac{1}{2i}(S_+ - S_-)\right)^2 + \alpha_3 S_z^2 \\ &= \tfrac{1}{4}\alpha_1(S_+^2 + S_-^2 + S_+S_- + S_-S_+) - \tfrac{1}{4}\alpha_2(S_+^2 + S_-^2 - S_+S_- - S_-S_+) + \alpha_3 S_z^2 \\ &= \tfrac{1}{4}(\alpha_1 - \alpha_2)(S_+^2 + S_-^2) + \tfrac{1}{4}(\alpha_1 + \alpha_2)(S_+S_- + S_-S_+) + \alpha_3 S_z^2 \end{aligned}$$

where

$$\begin{aligned} S_+S_- + S_-S_+ &= (S_x + iS_y)(S_x - iS_y) + (S_x - iS_y)(S_x + iS_y) \\ &= S_x^2 + S_y^2 - i[S_x, S_y] + S_x^2 + S_y^2 + i[S_x, S_y] \\ &= 2(S_x^2 + S_y^2) \\ &= 2(S^2 - S_z^2) \end{aligned}$$

$$\begin{aligned} \Rightarrow H_{int} &= \tfrac{1}{4}(\alpha_1 - \alpha_2)(S_+^2 + S_-^2) + \tfrac{1}{2}(\alpha_1 + \alpha_2)(S^2 - S_z^2) + \alpha_3 S_z^2 \\ &= \tfrac{1}{4}(\alpha_1 - \alpha_2)(S_+^2 + S_-^2) + \tfrac{1}{3}\left(\alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2)\right)(3S_z^2 - S^2) \\ &\qquad + \left(\tfrac{1}{3}\left(\alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2)\right) + \tfrac{1}{2}(\alpha_1 + \alpha_2)\right) S^2 \\ &= \tfrac{1}{4}(\alpha_1 - \alpha_2)(S_+^2 + S_-^2) + \tfrac{1}{3}\left(\alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2)\right)(3S_z^2 - S^2) \\ &\qquad + \tfrac{1}{3}(\alpha_1 + \alpha_2 + \alpha_3) S^2 \end{aligned}$$

(by using $\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right)_0 = 0$)

$$= \tfrac{1}{3}\left(\alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2)\right)(3S_z^2 - S^2) + \tfrac{1}{4}(\alpha_1 - \alpha_2)(S_+^2 + S_-^2)$$

$$\Rightarrow \begin{cases} A = \tfrac{1}{3}\left(\alpha_3 - \tfrac{1}{2}(\alpha_1 + \alpha_2)\right) \\ B = \tfrac{1}{4}(\alpha_1 - \alpha_2) \end{cases}$$

For spin-$\frac{3}{2}$,

$$S_z = \begin{bmatrix} \frac{3}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & \frac{3}{2} \end{bmatrix}, \quad S_+ = \begin{bmatrix} 0 & \sqrt{3} & & \\ & 0 & 2 & \\ & & 0 & \sqrt{3} \\ & & & 0 \end{bmatrix}, \quad S_- = \begin{bmatrix} 0 & & & \\ \sqrt{3} & 0 & & \\ & 2 & 0 & \\ & & \sqrt{3} & 0 \end{bmatrix}$$

$$S^2 = \tfrac{3}{2}\times\left(\tfrac{3}{2} + 1\right) I_4 = \tfrac{15}{4} I_4$$

$$\Rightarrow 3S_z^2 - S^2 = \begin{bmatrix} 3 & & & \\ & -3 & & \\ & & -3 & \\ & & & 3 \end{bmatrix}, \quad S_+^2 = \begin{bmatrix} 0 & 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 0 & 2\sqrt{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_-^2 = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 2\sqrt{3} & 0 & 0 & \\ 0 & 2\sqrt{3} & 0 & 0 \end{bmatrix}$$

$$A(3S_z^2 - S^2) + B(S_+^2 + S_-^2)$$

$$= \begin{bmatrix} 3A & 0 & 2\sqrt{3}B & 0 \\ 0 & -3A & 0 & 2\sqrt{3}B \\ 2\sqrt{3}B & 0 & -3A & 0 \\ 0 & 2\sqrt{3}B & 0 & 3A \end{bmatrix}$$

$$= \begin{bmatrix} 3A & 2\sqrt{3}B \\ 2\sqrt{3}B & -3A \end{bmatrix}_{spin\frac{3}{2} \oplus spin\frac{-1}{2}} \oplus \begin{bmatrix} -3A & 2\sqrt{3}B \\ 2\sqrt{3}B & 3A \end{bmatrix}_{spin\frac{1}{2} \oplus spin\frac{-3}{2}}$$

Eigenvalues : $\pm\sqrt{(3A)^2 + (2\sqrt{3}B)^2} = \pm\sqrt{9A^2 + 12B^2} = \pm\sqrt{3}\cdot\sqrt{3A^2+4B^2}$

$\lambda_+ = \sqrt{3}\cdot\sqrt{3A^2+4B^2}$.

two eigenvectors are

$$\left(|\tfrac{3}{2}\rangle \;\; |\tfrac{1}{2}\rangle\right)\frac{1}{N}\begin{bmatrix} 2\sqrt{3}B \\ \lambda_+ - 3A \end{bmatrix} \quad \text{and} \quad \left(|\tfrac{1}{2}\rangle \;\; |-\tfrac{3}{2}\rangle\right)\frac{1}{N'}\begin{bmatrix} 2\sqrt{3}B \\ \lambda_+ + 3A \end{bmatrix}$$

I. e.

$$\left(|\tfrac{3}{2}\rangle \;\; |\tfrac{1}{2}\rangle\right)\frac{1}{\sqrt{(\lambda_+ - 3A)^2 + 12B^2}}\begin{bmatrix} 2\sqrt{3}B \\ \lambda_+ - 3A \end{bmatrix} \quad \text{and} \quad \left(|\tfrac{1}{2}\rangle \;\; |-\tfrac{3}{2}\rangle\right)\frac{1}{\sqrt{(\lambda_+ + 3A)^2 + 12B^2}}\begin{bmatrix} 2\sqrt{3}B \\ \lambda_+ + 3A \end{bmatrix}$$

$\lambda_- = -\sqrt{3}\cdot\sqrt{3A^2+4B^2}$

two eigenvectors are

$$\left(|\tfrac{3}{2}\rangle \;\; |\tfrac{1}{2}\rangle\right)\frac{1}{\sqrt{(\lambda_- - 3A)^2 + 12B^2}}\begin{bmatrix} 2\sqrt{3}A \\ \lambda_- - 3A \end{bmatrix} \quad \text{and} \quad \left(|\tfrac{1}{2}\rangle \;\; |-\tfrac{3}{2}\rangle\right)\frac{1}{\sqrt{(\lambda_- + 3A)^2 + 12B^2}}\begin{bmatrix} 2\sqrt{3}B \\ \lambda_- + 3A \end{bmatrix}$$