

Lect 7. 1D harmonic oscillators — coherent states! ①

In this lecture we will use two different methods to solve the eigen-wavefunction and states of 1D harmonic oscillators.

§1. $H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m \omega^2 x^2$, define length unit $l = \sqrt{\frac{\hbar}{m\omega}}$

AMPAD $\Rightarrow H/\hbar\omega = -\frac{1}{2} \frac{d^2}{d(x/l)^2} + \frac{1}{2} (x/l)^2$

the eigen-equation $H\psi = E_n \psi_n \Rightarrow$

$$\left[-\frac{l^2}{2} \frac{d^2}{dx^2} + \frac{1}{2} \frac{x^2}{l^2} \right] \psi_n(x/l) = [E_n/\hbar\omega] \psi_n(x/l)$$

We first analyze the behavior of $\psi_n(x/l)$ at $x \rightarrow \pm\infty$. In this limit, we can neglect the constant $E_n/\hbar\omega$, we have

$$\left(-\frac{l^2}{2} \frac{d^2}{dx^2} + \frac{1}{2} \frac{x^2}{l^2} \right) \psi_n(x/l) \xrightarrow{x \rightarrow \pm\infty} 0$$

$$\Rightarrow \psi_n(x/l) \sim e^{\pm \frac{1}{2} \frac{x^2}{l^2}} \text{ at the leading order}$$

we need the normalization condition that $\int_{-\infty}^{+\infty} |\psi_n(x/l)|^2 dx$ finite, because all the states are bound states. we can only choose $\psi \sim e^{-\frac{1}{2} \frac{x^2}{l^2}}$.

Thus we try the solution

$$\psi_n = e^{-\frac{x^2}{2l^2}} u_n(x/l)$$

Plug this solution into the Schrödinger Eq, we have

$$\frac{d}{d(x/l)} \left[e^{-\frac{1}{2}(x/l)^2} u_n(x/l) \right] = -\frac{x}{l} e^{-\frac{1}{2}(x/l)^2} u_n(x/l) + e^{-\frac{1}{2}(x/l)^2} \frac{d}{d(x/l)} u_n(x/l)$$

$$\frac{d^2}{d(x/l)^2} \left[e^{-\frac{1}{2}(x/l)^2} u_n(x/l) \right] = (x/l)^2 e^{-\frac{1}{2}(x/l)^2} u_n(x/l) + e^{-\frac{x^2}{2l^2}} \frac{d^2}{d(x/l)^2} u_n - 2(x/l) e^{-\frac{x^2}{2l^2}} \frac{d}{d(x/l)} u_n - e^{-\frac{x^2}{2l^2}} u_n$$

$$\Rightarrow \frac{d^2}{d(x/l)^2} u_n - 2(x/l) \frac{d}{d(x/l)} u_n - \left(\frac{2E_n}{\hbar\omega} - 1 \right) u_n = 0$$

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define $z = x/l$, $\lambda_n = 2E_n/\hbar\omega \Rightarrow$

$$\boxed{\frac{d^2}{dz^2} u_n(z) - 2z \frac{d}{dz} u_n(z) - (\lambda_n - 1) u_n(z) = 0}$$

*) Some results quoted from the study of Hermite polynomials. only at $\lambda_n - 1 = 2n$, with "n" is a non-negative integers, we have polynomial solutions $H_n(z)$. The generation function for Hermite polynomials is

$$e^{-s^2 + 2zs} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} s^n$$

$$\Rightarrow H_n(z) = \frac{d^n}{ds^n} e^{-s^2 + 2zs} \Big|_{s=0} = e^{z^2} \frac{d^n}{ds^n} e^{-(s-z)^2} \Big|_{s=0} = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-(s-z)^2} \Big|_{s=0}$$

$$\boxed{H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} e^{-z^2}}$$

a few examples $H_n(z)$:

$H_0(z) = 1, H_1(z) = 2z, H_2(z) = 4z^2 - 2, H_3(z) = 8z^3 - 12z.$

They satisfy the relation:

they are normalized

$$\begin{cases} H_{n+1} - 2z H_n + H_{n-1} = 0 \\ \frac{d}{dz} H_n = 2n H_{n-1} \end{cases}$$

$$\int_{-\infty}^{+\infty} H_m(z) H_n(z) e^{-z^2} dz = \sqrt{\pi} 2^n n! \delta_{mn}$$

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⇒ normalized solution for $E_n = (n + 1/2) \hbar \omega,$

$$\psi_n(x) = \left[\frac{1}{\sqrt{\pi} 2^n n! l} \right]^{1/2} H_n\left(\frac{x}{l}\right) e^{-\frac{x^2}{2l^2}}$$

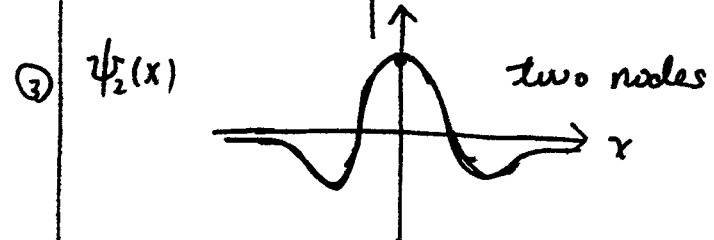
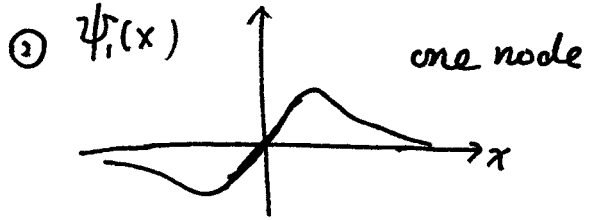
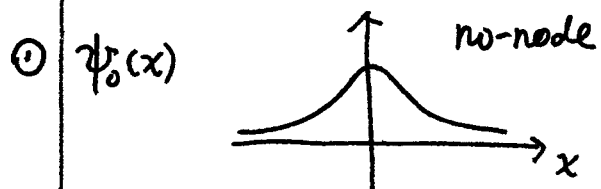
$\psi_0(x) = \frac{1}{\pi^{1/4} l^{1/2}} e^{-\frac{x^2}{2l^2}}$ even

$\psi_1(x) = \frac{\sqrt{2}}{\pi^{1/4} l^{1/2}} \frac{x}{l} e^{-\frac{x^2}{2l^2}}$ odd

$\psi_2(x) = \frac{1}{\pi^{1/4} \sqrt{2} l} \left[2\left(\frac{x}{l}\right)^2 - 1 \right] e^{-\frac{x^2}{2l^2}}$ even

$\psi_3(x) = \frac{1}{\pi^{1/4} \sqrt{3} l} \left[\frac{2}{3}\left(\frac{x}{l}\right)^2 - 1 \right] \left(\frac{x}{l}\right) e^{-\frac{x^2}{2l^2}}$ odd

Parity $\psi_n(-x) = (-1)^n \psi_n(x)$:

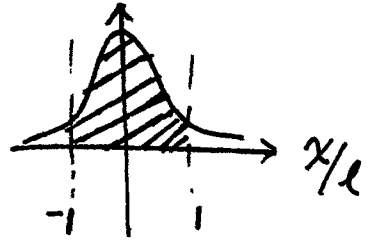


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Look at the ground state: Gaussian packet.

The classic region is at $|x/l| \leq 1$, the probability that the particle lying outside the classic region

$$\int_1^\infty e^{-z^2} dz / \int_0^\infty e^{-z^2} dz \approx 16\%$$



Algebraic solution

define $a = \frac{1}{\sqrt{2}} \left(\frac{x}{l} + \frac{ip l}{\hbar} \right)$, and $a^\dagger = \frac{1}{\sqrt{2}} \left(\frac{x}{l} - \frac{ip l}{\hbar} \right)$.

easy to check $[a, a^\dagger] = 1$.

Ex: please check $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = \frac{\hbar \omega}{2} [a a^\dagger + a^\dagger a] = \hbar \omega [a^\dagger a + 1/2]$

, $a^\dagger a$ is called the phonon number operator.

Please note: we have changed our viewpoint. Before, we viewed

$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$ as a single particle problem with many eigenmodes, and the particle is not in the free space. But now, we rewrite $H = \hbar \omega [a^\dagger a + 1/2]$

it becomes a single-mode phonon problem, and the number of

phonons

(n) is related to the n-th excited state.

Now let us solve the spectra of $a^\dagger a$. First, $a^\dagger a$ is non-negative.

① For any state $|\psi\rangle$, $\langle \psi | a^\dagger a | \psi \rangle = |a|\psi\rangle|^2 \geq 0$, thus all the eigenvalues of $a^\dagger a$ should be non-negative.

⑦ Suppose $a^\dagger a |n\rangle = n |n\rangle$, where n is eigenvalue with $n \geq 0$.

please check $[a^\dagger a, a^\dagger] = a^\dagger$, $[a^\dagger a, a] = -a$

$\Rightarrow a^\dagger a (a|n\rangle) = (n-1) (a|n\rangle)$

thus $a|n\rangle$ is also $a^\dagger a$'s eigenstate, with the eigenvalue $n-1$.

thus we have a series of eigenstates

$|n\rangle, a|n\rangle, a^2|n\rangle, \dots$, with eigenvalues $n, n-1, n-2, \dots$.

Thus n has to be an integer number, such that this sequence has to be terminated at $n=0$. we have $|0\rangle$, but $a|0\rangle = 0$.

Now we start from $|0\rangle$, and apply a^\dagger successively, then we arrive at the sequence

$|0\rangle, a^\dagger|0\rangle, (a^\dagger)^2|0\rangle, \dots$, We define $|n\rangle = N_n (a^\dagger)^n |0\rangle$,

where N_n is the normalization factor,

$(a^\dagger a) a n\rangle = (n-1) a n\rangle$ $(a^\dagger a) a^\dagger n\rangle = (n+1) a^\dagger n\rangle$	such that $\langle n n\rangle = 1$. ← please prove.
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now, we determine N_n .

$$\langle n|n\rangle = \left| \frac{N_n}{N_{n-1}} \right|^2 \langle n-1| a a^\dagger |n-1\rangle = \left| \frac{N_n}{N_{n-1}} \right|^2 \langle n-1| (a^\dagger a + 1) |n-1\rangle$$

$$= n \left| \frac{N_n}{N_{n-1}} \right|^2 = 1$$

we can choose N_n to be real $\Rightarrow N_n = \sqrt{n} N_{n-1}$.

with the definition $\Rightarrow N_0 = 1. \Rightarrow N_n = \sqrt{n!}$

$$\Rightarrow |n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle.$$

AMPAL Ex: 1) Prove the matrix elements

$$\langle m | x | n \rangle = \frac{l}{\sqrt{2}} (\sqrt{n+1} \delta_{m,n+1} + \sqrt{n} \delta_{m,n-1})$$

$$\langle m | p | n \rangle = \frac{i\hbar}{\sqrt{2}l} (\sqrt{n+1} \delta_{m,n-1} - \sqrt{n} \delta_{m,n+1})$$

2) Check $[x, p] = i\hbar$, using the above matrix elements

3) prove that for the state $|n\rangle$,

$$\langle n | x^2 | n \rangle = l^2 (n + \frac{1}{2}) \quad \langle n | p^2 | n \rangle = \frac{\hbar^2}{l^2} (n + \frac{1}{2})$$

$$\Rightarrow \Delta x \cdot \Delta p = (n + \frac{1}{2}) \hbar \quad \leftarrow \text{at } n=0, \text{ the uncertain relation reaches the minimum.}$$

* Now we need to work out the wavefunction.

For ground states $|0\rangle$, we have $a|0\rangle = 0$.

$$\langle x | a | 0 \rangle = \langle x | \left(\frac{\hat{x}}{l} + i\hat{p} \right) | 0 \rangle = \left[\frac{x}{l} + \frac{\partial}{\partial x} \cdot l \right] \langle x | 0 \rangle = 0$$

$$\Rightarrow \psi_0(x) = \langle x | 0 \rangle = \frac{1}{\pi^{1/4} l^{1/2}} e^{-\frac{x^2}{2l^2}}$$

and

$$\psi_n(x) = \frac{1}{\pi^{1/4} l^{1/2}} \frac{1}{\sqrt{n!}} \left(\frac{x}{l} - l \frac{d}{dx} \right)^n e^{-\frac{x^2}{2l^2}}$$

*) Coherent states

We first prove Baker-Hausdorff lemma: $e^{i\lambda G} A e^{-i\lambda G} = A + i\lambda [G, A] + \frac{(i\lambda)^2}{2!} [G, [G, A]] + \dots + \frac{(i\lambda)^n}{n!} [G, [G, \dots [G, A] \dots]] + \dots$

Proof: define $O(\lambda) = e^{i\lambda G} A e^{-i\lambda G}$, $\Rightarrow \frac{d}{d\lambda} O = i e^{i\lambda G} [G, A] e^{-i\lambda G} = i [G, O(\lambda)]$

$\Rightarrow O(\lambda) = O(0) + i \int_0^\lambda d\lambda_1 [G, O(\lambda_1)]$, and $O(0) = A$

$\Rightarrow O(\lambda) = A + i \int_0^\lambda d\lambda_1 [G, A] + i \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 [G, [G, O(\lambda_2)]]$
 $= A + i \int_0^\lambda d\lambda_1 [G, A] + \dots + i^n \int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \dots \int_0^{\lambda_{n-1}} d\lambda_n [G [G, \dots [G, A] \dots]] + \dots$

plug in $\int_0^\lambda d\lambda_1 \int_0^{\lambda_1} d\lambda_2 \dots \int_0^{\lambda_{n-1}} d\lambda_n = \frac{\lambda^n}{n!}$, we arrive at the above lemma.

or $e^B A e^{-B} = A + [B, A] + \frac{1}{2!} [B, [B, A]] + \dots + \frac{1}{n!} [B, [B, \dots [B, A] \dots]] + \dots$

Define coherent states as eigenstates of a , i.e. $a|\alpha\rangle = \alpha|\alpha\rangle$ where α can be a complex number.

using Baker-Hausdorff lemma, $e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} = a - [\alpha a^\dagger, a] = a + \alpha$

$\Rightarrow e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} |0\rangle = \alpha |0\rangle \Rightarrow a (e^{\alpha a^\dagger} |0\rangle) = \alpha (e^{\alpha a^\dagger} |0\rangle)$

$\Rightarrow |\alpha\rangle = N_\alpha e^{\alpha a^\dagger} |0\rangle$

$$\langle \alpha | \alpha \rangle = |N_\alpha|^2 \langle 0 | e^{\frac{\alpha^* a}{\alpha a}} e^{-\frac{\alpha a}{\alpha a}} | 0 \rangle$$

using Baker-Hausdorff: we prove $e^{\alpha^* a} e^{\alpha a} e^{-\alpha^* a} = ?$

$$e^B e^A e^{-B} = e^A + [B, e^A] + \dots + \frac{1}{n!} [B, \dots [B, e^A]] + \dots$$

now $B = \alpha^* a, e^A = e^{\alpha a} \Rightarrow [B, e^A] = \alpha^* [a, e^{\alpha a}]$

according to $e^{-\alpha a} a e^{\alpha a} = a + \alpha \Rightarrow a e^{\alpha a} = e^{\alpha a} a + \alpha e^{\alpha a}$

$$\Rightarrow [B, e^A] = \alpha^* [a, e^{\alpha a}] = \alpha^* \alpha e^{\alpha a} = \alpha^* \alpha e^A$$

$$\Rightarrow e^B e^A e^{-B} = e^A \sum_{n=0}^{\infty} \frac{(\alpha^* \alpha)^n}{n!} = e^A e^{|\alpha|^2} \Rightarrow e^{\alpha^* a} e^{\alpha a} = e^{\alpha a} e^{\alpha^* a} e^{|\alpha|^2}$$

$$\Rightarrow \langle \alpha | \alpha \rangle = |N_\alpha|^2 e^{|\alpha|^2} \langle 0 | e^{\alpha a} e^{\alpha^* a} | 0 \rangle = |N_\alpha|^2 e^{|\alpha|^2} = 1$$

$$\Rightarrow \text{normalization factor } N_\alpha = e^{-|\alpha|^2/2}$$

$$\Rightarrow |\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle$$

$$e^A e^B = e^B e^A e^{[A,B]}$$

$$= e^{A+B} e^{\frac{1}{2}[A,B]}$$

if $[A,B]$ commutes with A , and B .

Ex: Please use the fact of $a|\alpha\rangle = \alpha|\alpha\rangle$, prove that for state

$$|\alpha\rangle, \text{ define } \overline{\Delta X^2} = \langle \alpha | X^2 | \alpha \rangle - (\langle \alpha | X | \alpha \rangle)^2$$

$$\text{and } \overline{\Delta p^2} = \langle \alpha | p^2 | \alpha \rangle - (\langle \alpha | p | \alpha \rangle)^2$$

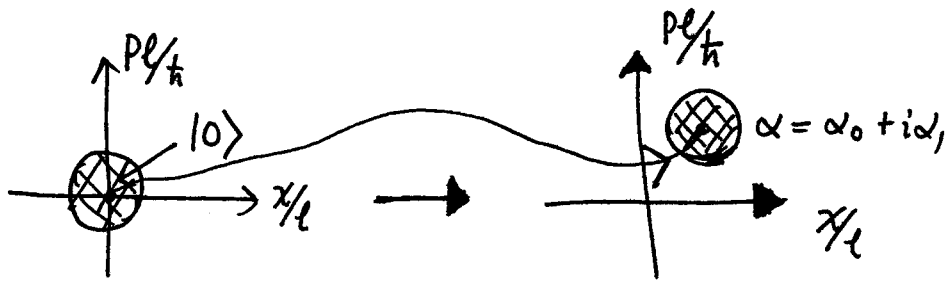
$$\text{we have } \sqrt{\overline{\Delta X^2}} \sqrt{\overline{\Delta p^2}} = \frac{\hbar}{2}$$

What's the physical meaning of $|\alpha\rangle$, define $\alpha = \alpha_0 + i\alpha_1$

$$\Rightarrow \left[\frac{(x - l\alpha_0)}{l} + l \left(\frac{\partial}{\partial x} - i \frac{\alpha_1}{l} \right) \right] \psi_\alpha(x) = 0$$

$$\Rightarrow \psi_\alpha(x) = \frac{1}{\pi^{1/4} l^{1/2}} e^{-\frac{(x - l\alpha_0)^2}{2l^2} + i \frac{\alpha_1}{l} x}$$

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* equation of motion in the Heisenberg picture.

$$a^\dagger(t) = e^{iHt} a^\dagger e^{-iHt}$$

$$a(t) = e^{iHt} a e^{-iHt}$$

$$= e^{i\omega t} a^\dagger e^{-i\omega t}$$

$$= e^{i\omega t} a e^{-i\omega t}$$

$$\Rightarrow \frac{d}{dt} a^\dagger(t) = i\omega a^\dagger$$

$$a^\dagger(t) = a^\dagger e^{i\omega t}$$

$$\frac{d}{dt} a(t) = -i\omega a$$

$$a(t) = a e^{-i\omega t}$$

$$\Rightarrow x(t) = \frac{l}{\sqrt{2}} [a^\dagger(t) + a(t)] = \frac{l}{\sqrt{2}} [a^\dagger(0)e^{i\omega t} + a(0)e^{-i\omega t}]$$

$$= x(0) \cos \omega t + \frac{p(0)}{m\omega} \sin \omega t$$

$$p(t) = \frac{\hbar}{\sqrt{2}i} [a^\dagger(t) - a(t)] = -m\omega x(0) \sin \omega t + p(0) \cos \omega t !$$

(*) $U(1)$ or $SO(2)$ symmetry in phase space

Harmonic oscillator Hamiltonian is invariant, under the transformation

$$\begin{cases} a^\dagger \rightarrow a^\dagger e^{i\theta} \\ a \rightarrow a e^{-i\theta} \end{cases} \quad \text{or} \quad \begin{pmatrix} x'/\ell \\ \frac{p'}{\hbar} \end{pmatrix} = \begin{pmatrix} \frac{x}{\ell} \cos\theta + \frac{p}{\hbar} \sin\theta \\ -\frac{x}{\ell} \sin\theta + \frac{p}{\hbar} \cos\theta \end{pmatrix}$$

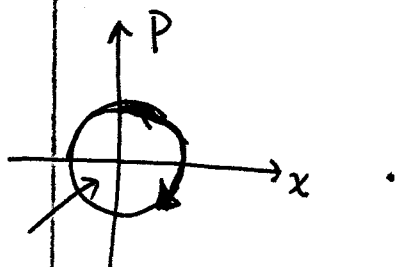
This transformation can be generated by $R = e^{-i\theta a^\dagger a}$

i.e. $R^\dagger(\theta) a^\dagger R = e^{i\theta a^\dagger a} a^\dagger e^{-i\theta a^\dagger a} = e^{i\theta} a^\dagger$

The generator of this $U(1)$ symmetry is nothing but the Hamiltonian!
 proportional

This is an angular momentum in phase-space: (but x - p conjugate)

$$\begin{aligned} L_{\text{phase-space}} &= x \pi_p - p \pi_x, \quad \text{according to } \pi_x = p \\ &= -(x^2 + p^2) \text{ which is non-negative!} \quad \pi_p = -x. \end{aligned}$$



harmonic oscillator's motion in phase space is chiral!!

Compare with the case of usual 2D

$$\begin{aligned} \Rightarrow L_z &= x p_y - y p_x \\ &= -i\hbar \frac{\partial}{\partial \phi} \end{aligned}$$

L_z 's spectra $-\infty, \dots, -1, 0, 1, \dots, +\infty$

take all the integer values.

(no-chiral)

What's the symmetry of the 2D, or more generally nD harmonic oscillator?

$$H = \hbar\omega \left[\frac{n}{2} + (a_1^\dagger, \dots, a_n^\dagger) \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right] \quad (\text{quadratic hamiltonians can be variable-separated!})$$

introducing Unitary transformation $\begin{pmatrix} a'_1 \\ \vdots \\ a'_n \end{pmatrix} = U \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \Rightarrow$ The hamiltonian is invariant,

i.e. the symmetry is $U(n)$, not just $SO(n)$. (You can take $n=2$ or 3 as examples).

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The degeneracy pattern for n-Dimensional harmonic oscillator,

$$n_1 + n_2 + \dots + n_n = m, \quad \text{where } n_i \text{ is the quantum number along } i\text{-th direction,}$$

\rightarrow number of degeneracy $0|0|00$
 $g_n(m) = \binom{m+n-1}{n-1} = \frac{(m+n-1)!}{m!(n-1)!}$
 m - balls
 n-1 - baffles

n_i is non-negative integer.

① 1D : $g_1(m) = 1$

③ 3D : $g_3(m) = \frac{(m+2)!}{m!2!} = \frac{(m+2)(m+1)}{2}$

② 2D : $g_2(m) = m+1$

more formally, the m-th level of nD harmonic oscillator

belongs to the  representation of $SU(n)$ group.

$(m=0, 1, 2, \dots)$

Q: What are the generators of the $U(n)$ transformation? Let's only take the 2D case as an example.

The rotation in real space $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} p'_x \\ p'_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$

Such a rotation is generated by the usual angular momentum L_{xy} ,

the rotation in $(\begin{smallmatrix} x-y \\ P_x-P_y \end{smallmatrix})$ planes, i.e. $L_{xy} = \frac{xP_y - yP_x}{\hbar} = -i(a_x^\dagger a_y - a_y^\dagger a_x)$

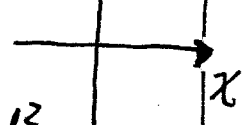
The rotation operator is $R_{xy}(\theta) = e^{-iL_{xy}\theta}$.

For harmonic potential $\frac{P_y^2}{2m} + \frac{1}{2}m\omega^2 y^2$, we can define a canonical transformation $(\frac{y}{\ell}, \frac{\ell P_y}{\hbar}) \rightarrow (-\frac{\ell P_y}{\hbar}, \frac{y}{\ell})$.

or we define rotation in the $x \leftrightarrow -\frac{P_y \ell^2}{\hbar}$; $P_x - \frac{\hbar y}{\ell^2}$ planes

as $\begin{pmatrix} x' \\ -\frac{P_y \ell^2}{\hbar} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ -\frac{P_y \ell^2}{\hbar} \end{pmatrix}$ and $\begin{pmatrix} P_x' \\ \frac{\hbar y}{\ell} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} P_x \\ \frac{\hbar y}{\ell} \end{pmatrix}$

This transformation is generated in the plane: $x - \frac{P_y \ell^2}{\hbar}$.



$$Q_{xy} = \frac{1}{\hbar} \left[x \cdot \frac{\hbar y}{\ell^2} - \left(-\frac{P_y \ell^2}{\hbar} \right) P_x \right] = \frac{xy}{\ell^2} + \frac{P_x P_y \ell^2}{\hbar^2} = (a_x^\dagger a_y + a_y^\dagger a_x)$$

Similarly, we can have the rotations in $(x, P_x \ell^2 / \hbar)$ and $(y, P_y \ell^2 / \hbar)$

planes. $Q_{xx} = a_x^\dagger a_x$, $Q_{yy} = a_y^\dagger a_y$

we decompose $U(2) = U(1) \otimes SU(2)$

$a_x^\dagger a_x + a_y^\dagger a_y$ $\begin{cases} \frac{1}{2} (a_x^\dagger a_x - a_y^\dagger a_y) \\ \frac{1}{2} (a_x^\dagger a_y \pm a_y^\dagger a_x) \end{cases}$