Lect 10 Superconductivity

- The superfluidity theory we have discussed is for neutral systems. How about for charged systems, say, superconductors?

The neutral boson Lagrangian

$$\mathcal{L}(\varphi) = \frac{i}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) - \frac{1}{2m} \partial_x \varphi^* \partial_x \varphi + \mu |\varphi|^2 - \frac{V_0}{2} |\varphi|^4$$

has the $U(1)$ symmetry $\varphi \to e^{if}\varphi$, but it does not have the global symmetry $\varphi(x t) \to e^{if(x t)}\,\varphi(x,t)$.

For charged bosons (Cooper pairs of electrons)

- $$\mathcal{L}(\varphi, A_\mu) = \frac{i}{2}\left(\varphi^*(\partial_0 + iA_0)\varphi - \varphi(\partial_0 - iA_0)\varphi^*\right) - \frac{1}{2m}\left|(\partial_i + iA_i)\varphi\right|^2$$

$$+ \mu|\varphi|^2 - \frac{V_0}{2}|\varphi|^4 + \frac{1}{8\pi e^2}\left(\frac{1}{c}E^2 - cB^2\right)$$

where $E_i = \partial_0 A_i - \partial_i A_0 = F_{0i}$, $B_i = \epsilon_{ijk}\partial_j A_k = \frac{1}{2}\epsilon_{ijk}F_{jk}$

which is invariant as

$$\varphi \to \tilde{\varphi} = e^{if(x,t)}\varphi, \quad \tilde{A}_\mu = A_\mu - \partial_\mu f$$

If φ_c, $\cancel{A}$ satisfy the classical equation of motion (fix A_μ).

$\Rightarrow$ do variation respected to $\delta\varphi$: $\varphi' = e^{if(x,t)}\varphi \Rightarrow \delta\varphi' = \varphi(1 + if(x t))$

$$J_i = \frac{-i}{2m}(\varphi^* \partial_i \varphi - \partial_i \varphi^* \cdot \varphi) + A_i|\varphi|^2$$

- $\Rightarrow \mathcal{L}[\tilde{\varphi}, A_\mu] = \mathcal{L}[\varphi, A_\mu - \partial_\mu f]$

$\Rightarrow \delta\mathcal{L} = \int d^d x\, dt\; \partial_\mu f \cdot J^\mu \quad \Rightarrow \quad \partial_\mu J^\mu = 0$ where $J_0 = \varphi^*\varphi$,

* Current correlation function & E-M responses

$$\mathcal{L}[\varphi, A_\mu] = \mathcal{L}[\varphi] - A_0 j^0 - A_i j^i - \frac{1}{2m}(A_i)^2 \rho$$

we use the linear response theory $\qquad j_i = \frac{-i}{2m}[\varphi^*(\partial_i\varphi) - (\partial_i\varphi^*)\varphi]$

$$\langle J^\mu(x,t)\rangle = \langle j^\mu(x,t)\rangle + (1-\delta_{\mu,0}) A_\mu \rho$$

$$= \int d^d x\, dt \;\; \Pi^{\mu\nu}(xt; x't')\, A_\nu(x',t')$$

The response function ~~read~~ reads

$$\Pi^{0\nu}(xt; x't') = -i\theta(t-t')\langle[\rho(x,t), \rho(x',t')]\rangle$$

$$\Pi^{0i}(xt; x't') = -i\theta(t-t')\langle[\rho(xt), j^i(x't')]\rangle$$

$$\Pi^{i0}(xt; x't') = -i\theta(t-t')\langle[j^i(xt)\;\; \rho(xt')]\rangle$$

$$\Pi^{ij}(xt; x't') = -i\theta(t-t')\langle[j^i(xt), j^j(x't')]\rangle + \delta^{ij}\delta(x-x')\delta(t-t')\frac{\langle\rho\rangle}{m}$$

$\Pi^{\mu\nu}$ satisfies $\quad \boxed{(\Pi^{\mu\nu}(xt; x't'))^* = \Pi^{\nu\mu}(x't'; xt)}$

$$\Rightarrow (\Pi^{\mu\nu}(k))^* = \Pi^{\nu\mu}(-k)$$

$\Pi^{\mu\nu}$ satisfies continuity & gauge invariance conditions.

$$\partial_\mu J^\mu = 0 \quad \Rightarrow \quad \boxed{k_\mu \Pi^{\mu\nu}(k) = 0}$$

$$A \to A + \partial f \quad \to \quad \boxed{\Pi^{\mu\nu}(k)\, k_\nu = 0}$$

we can decompose $\Pi^{ij}(k)$ into transverse & longitudinal parts

spatial part $\rightarrow$ $$\Pi^{ij}(k) = \frac{k_i k_j}{k^2}\Pi^{\parallel}(k) + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Pi^{\perp}(k)$$

$$\Pi^{0i}(k) = \left(\Pi^{i0}(-k)\right)^{\dagger} = -\frac{k_j}{\omega}\Pi^{ji}(k) \Leftarrow k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\Pi^{00}(k) = -\frac{k_j}{\omega}\Pi^{0j} = \frac{k_i k_j}{\omega^2}\Pi^{ij}(k)$$

$\Rightarrow$ $$\boxed{\begin{aligned} \Pi^{0i}(k) &= -k_i \Pi^{\parallel}(k) \\ \Pi^{00}(k) &= \frac{k^2}{\omega^2}\Pi^{\parallel}(k). \end{aligned}}$$

the time component is only related to the ~~longitudinal~~ longitudinal component.

For the correlation in the spatial diretion, we define the paramagnetic contribution as

$$\Pi^{ij}_{para}(xt;x't') = -i\Theta(t-t')\langle [j^i(xt), j^j(x't')]\rangle,$$

then $$\Pi^{\parallel}(k) = \Pi^{\parallel}_{para}(k) + \frac{\langle\rho\rangle}{m}$$

$$\Pi^{\perp}(k) = \Pi^{\perp}_{para}(k) + \frac{\langle\rho\rangle}{m}$$

diamagnetic Contribution.

we know $\delta\rho(k) = \Pi^{00}(k)\, A_0(k)$,

Since $A_0(k)$ is just external potential, thus $-\Pi^{00}(k)$ is just the Compressbility $\chi(k) = -\Pi^{00}(k)$. Let us assume that $\chi(k) \rightarrow$ Const ($k\rightarrow$

then $\boxed{\Pi^{\parallel}(k) = -\chi(k)\frac{\omega^2}{k^2}}$, thus at the static limit ($\omega \to 0$ first; $k \to 0$ second), $\Pi^{\parallel}(k)$ goes to zero (longitudinal field has no effect!), thus $\Pi^{\parallel}_{para}(k)$ must exactly ~~can~~ cancel the diamagnetic contribution.

On the other hand, if $k \to 0$ first and $\omega \to 0$ second $\boxed{(k \ll \omega)}$.

$-i\omega A(k,\omega)$ is the electric field

$$J^i(\omega) = \lim_{k\to 0} \frac{\Pi^{ij}}{-i\omega} \underbrace{(-i\omega) A_j(k,\omega)}_{E(\omega,k)}$$

$$\Rightarrow \quad \sigma(\omega) = \frac{\Pi^{\parallel}(\omega,0)}{-i\omega} = \frac{\Pi^{\perp}(\omega,0)}{-i\omega}$$

in the normal state, as $k \to 0$, there's no difference between transverse and longitudinal modes.

$\nabla\cdot\vec{D} = 4\pi\rho \Rightarrow \vec{D} = \vec{E} + 4\pi\vec{P} = (1+4\pi\chi)\vec{E} = \epsilon\vec{E}$

$\vec{j} = \partial_t P \Rightarrow j(\omega) = -i\omega P(\omega) = -i\omega\chi(\omega)\vec{E}$

$\vec{j}(\omega) = \frac{-i}{4\pi}(\epsilon - 1)\omega\, \vec{E}(\omega)$

i.e. $\sigma(\omega) = \frac{-i}{4\pi}(\epsilon - 1)\omega$ i.e. $\boxed{\epsilon(\omega) = 1 + \frac{i 4\pi\sigma(\omega)}{\omega}}$

how about in the static limit $\omega << k$, $\Pi^{\perp}(k,\omega)$?

From $\nabla \times \vec{M} = \vec{j}$

$$\Rightarrow j_i = i\,\epsilon^{ijk} k_j M_k = -\Pi^{ij} A_k = -(\delta_{ij} k^2 - k^i k^j) A_j \frac{\Pi^{\perp}}{k^2}$$

$$= \epsilon^{ij'k'} k_{j'} \epsilon^{k'i'j} k_{i'} A_j \frac{\Pi^{\perp}(k,\omega)}{k^2} = -i\,\epsilon^{ijk} k_j B_k \frac{\Pi^{\perp}(k,\omega)}{k^2}$$

$$\Rightarrow \boxed{M_i = -\frac{\Pi^{\perp}(k,0)}{k^2} B_i}$$, i.e. $\frac{\Pi^{\perp}(k,0)}{k^2}$ is magnetic susceptibility.

Summarize: Compressibility $\lim_{k\to 0} \chi_c(k) = -\lim_{k\to 0} k^2 \lim_{\omega\to 0} \frac{\Pi^{\parallel}(k,\omega)}{\omega^2}$

orbital magnetic susceptibility $\lim_{k\to 0} \chi_m(k) = -\lim_{k\to 0} \frac{1}{k^2} \Pi^{\perp}(k,0)$

* E-M:

let us calculate $\Pi^{\mu\nu}$ for bosonic field:

$$\mathcal{L} = \frac{\chi}{2}\left((\partial_0 \theta + A_0)^2 - v^2 (\partial_i \theta + A_i)^2 \right)$$

the paramagnetic current $j_0 = -\chi \partial_0 \theta$, $j_i = \chi v^2 \partial_i \theta$

total current $J_0 = -\chi(\partial_0 \theta + A_0)$ $J_i = \chi v^2 (\partial_i \theta + A_i)$

$$\Pi^{00}_{para}(k,\omega) = \chi^2 (-i\omega)(+i\omega)\langle \Theta(\omega k)\Theta(-\omega,-k)\rangle = \frac{\chi^{\cancel{4}}\omega^2}{\omega^2 - v^2k^2 + i0^+ \mathrm{sgn}(\omega)}$$

$\hookrightarrow$ retarded green's function

$$\Pi^{0i}_{para}(k,\omega) = \Pi^{i,0}_{para}(k,\omega)$$

$$= \frac{\chi^2}{v^2}(-i\omega)(-ik_i)\langle \Theta(\omega,k)\,\Theta(-\omega,-k)\rangle = \frac{v^2\chi^{\cancel{4}}(-)k_i\omega}{\omega^2 - v^2k^2 + i0^+ \mathrm{sgn}(\omega)}$$

$$\Pi^{ij} = \chi^2(ik_i)(-ik_j)\langle \Theta(\omega k)\Theta(-\omega,-k)\rangle = \chi v^4 \frac{k_i k_j}{\omega^2 - v^2k^2 + i0^+ \mathrm{sgn}(\omega)}$$

$$\Rightarrow \Pi^{00}(k,\omega) = \chi\left[\frac{\omega^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)} - 1\right] = +\chi\left[\frac{v^2k^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}\right]$$

$$\Pi^{0i}(k,\omega) = \Pi^{i0}(k,\omega) = -\chi v^2 \frac{\omega k_i}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}$$

$$\Pi^{ij}(k,\omega) = \chi v^2\left(\delta_{ij} + v^2\frac{k_i k_j}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}\right)$$

$$\Rightarrow \boxed{\begin{aligned} \Pi^{\parallel} &= \chi v^2\left(1 + \frac{v^2k^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}\right) = \chi v^2 \frac{\omega^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)} \\ \Pi^{\perp} &= \chi v^2 \end{aligned}}$$

$\Rightarrow$ the Compressbility $\lim_{k\to 0}\lim_{\omega\to 0} -\Pi^{00}(k,\omega) = \lim_{k\to 0} -\Pi^{00}(k,0) = \chi.$

magnetic susceptibility $-\lim_{k\to 0}\frac{1}{k^2}\Pi^{\perp}(k,0)$ divergences.

~~real part of~~ if we chose transverse gauge $\partial_i A_i = 0$

$$\Rightarrow \boxed{J^i = \Pi^{ij}A_j = \Pi^{\perp}A_i = \chi v^2 A_i = +\frac{\rho}{m}A_i}$$

London equation!

optical conductivity $\boxed{\mathrm{Re}\,\sigma(\omega) = -\mathrm{Im}\frac{\Pi''(\omega,0)}{\omega} = \mathrm{Im}\frac{\chi v^2}{\omega + i0^+} = \frac{\pi\rho}{m}\delta(\omega)}$

which is zero at finite frequency.

* Anderson - Higgs mechnism

In charged superfluid (superconductor), the gapless Goldstone mode is gone. The gauge field A_μ has its own dynamics.

$$Z = \int D\varphi DA_\mu \; e^{i\int d^dx\, dt \; \mathcal{L}(\varphi, A_\mu) + \frac{1}{8\pi e^2}(E^2 - B^2)}$$

in the symmetry breaking ground states $\varphi = |\varphi| e^{i\theta(x,t)}$

we can absorb the phase by doing a gauge transformation

$$\left|(\partial_\mu - iA_\mu)|\varphi|e^{i\theta(x,t)}\right|^2 = \left|(\partial_\mu \underbrace{-iA_\mu - i\partial_\mu\theta}_{i\tilde{A}_\mu})|\varphi|\right|^2$$

$i\tilde{A}_\mu$ which is equavilant to iA_μ, and does not give new physical field configuration.

thus φ can be made real:

$$\varphi = \bar{\varphi} + \delta\varphi$$

$\Rightarrow$

~~to~~ or equivalently we can set $\theta = 0$ in the action

$$\mathcal{L} = \frac{\chi}{2}\left[(\partial_0\theta + A_0)^2 - v^2(\partial_i\theta + A_i)^2\right] = \frac{1}{2V_0}A_0^2 - \frac{\rho}{2m}A_i^2$$

$\Rightarrow$ the Langrangian for A becames

$$\mathcal{L} = \frac{1}{2V_0} A_0^2 - \frac{\rho}{2m} A_i^2 + \frac{1}{8\pi e^2} \left(E^2 - B^2 \right)$$

note that $\frac{1}{8\pi e^2} E^2 = \frac{1}{8\pi e^2} [\partial_0 A_i]^2 + \frac{1}{8\pi e^2} (\partial_i A_0)^2 - \frac{1}{4\pi e^2} \partial_i A_0 \partial_0 A_i$

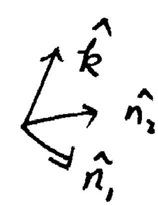
and thus A_0 has no time derivative. Let's integrate out A_0.

$$A_0 \Big(\frac{1}{2V_0} - \underbrace{\frac{\partial_i^2}{8\pi e^2}}_{\text{neglected}} \Big) A_0 + \frac{1}{4\pi e^2} A_0 \, \partial_0 \partial_i A_i$$

$$\approx \underbrace{\frac{1}{2V_0} \Big(A_0 - \frac{V_0}{4\pi e^2} \partial_0 \partial_i A_i \Big)^2}_{\text{const after gaussian integration}} - \frac{1}{2V_0} \frac{V_0^2}{(4\pi e^2)^2} (\partial_0 \partial_i A_i)(\partial_0 \partial_j A_j)$$

$$\Rightarrow \mathcal{L}_{eff} = \partial_0 A_j \Big[\frac{1}{8\pi e^2} \delta_{ij} + \frac{1}{2} \frac{V_0}{(4\pi)^2 e^4} \underbrace{\partial_i \partial_j}_{\text{longitudinal}} \Big] \partial_0 A_i - \frac{\rho}{2m} A_i^2 - \underbrace{\frac{B^2}{8\pi e^2}}_{\text{transverse}}$$

Let us decompose $A_i = \hat{k}_i A^{\parallel} + \hat{n}_a A_a^{\perp}$



$$\mathcal{L}_{eff} = \frac{1}{8\pi e^2} \left[(\partial_0 A_a^{\perp})^2 - (\partial_i A_a^{\perp})^2 \right] - \frac{\rho}{2m} (A_a^{\perp})^2$$

$$+ \frac{1}{8\pi e^2} \partial_0 A^{\parallel} \Big(1 + \frac{V_0}{4\pi e^2} (\partial_i)^2 \Big) \partial_0 A^{\parallel} - \frac{\rho}{2m} (A^{\parallel})^2$$

all the 3 modes has the gap $\Delta = e\sqrt{4\pi \rho / m}$

replace $\partial_0^2 = -\Delta^2$ and $v^2 = V_0 \rho / m$

$$\Rightarrow \mathcal{L}_{eff} = \frac{1}{8\pi e^2} \left[(\partial_0 A_a^{\perp})^2 - (\partial_i A_a^{\perp})^2 \right] - \frac{\rho}{2m} (A_a^{\perp})^2 + \frac{1}{8\pi e^2} \left[(\partial_0 A^{\parallel})^2 - v^2 (\partial_i A^{\parallel})^2 \right] - \frac{\rho}{2m} (A^{\parallel})^2$$

* Superfluidity & superfluid density:

let's consider a boson system which is invariant under Galileo transformation, and assum an excitation spectrum $\mathcal{E}(k)$. Consider a single excitation $\mathcal{E}.(k)$, thus $\vec{P}=\vec{k}$ and $E = E_{ground} + \mathcal{E}$. Let us boost the system by velocity $\vec{v}$, ~~excitation has a doppler shift~~ the total energy and momentum change to

$$E = E_{ground} + \epsilon + \frac{1}{2} N m v^2 + \vec{v}\cdot\vec{k}$$

$$\vec{P} = \vec{k} + N m \vec{v}$$

Thus compared with boosted ground state, we have

$$\mathcal{E}_v(k) = \mathcal{E}(k) + \vec{k}\cdot\vec{v}, \quad \vec{P} = \vec{k}.$$

~~In the equilibrium~~

In the boosted Superfluid, in the equilibrium, we have distribution according to $\mathcal{E}_v(k)$.

$$E_v = E_g + \sum_k \mathcal{E}_v(k)\, n_B(\mathcal{E}_v(k)) + \frac{1}{2} N m v^2$$

$$\vec{P} = N m \vec{v} + \sum_k \vec{k}\, n_B(\mathcal{E}_v(\vec{k}))$$

The momentum can be written as, (expand to first order of $\vec{v}\cdot\vec{k}$).

$$\vec{P} = (Nm - V \rho_n m)\, \vec{v} = V \underline{\rho_s} m \vec{v}$$

↑ superfluid density.

where $\rho_n = -\frac{1}{md}\int\frac{d^dk}{(2\pi)^d}\, k^2\, n_B'(\varepsilon(k))$.

$\uparrow$ from the angle integral $\vec{v}\cdot\vec{k}$

average

If $\varepsilon(k)\propto|k|$

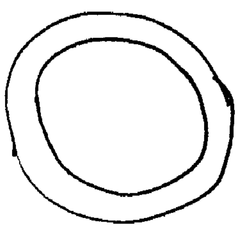
$\Rightarrow \quad \rho_n \propto T^{d+1}$.

Another question is: Since super-current carring state is not the lowest energy state, why it is stable ?

Let use consider a symmetry breaking state $\varphi = \varphi_0 + \delta\varphi$

condensate fluctuation

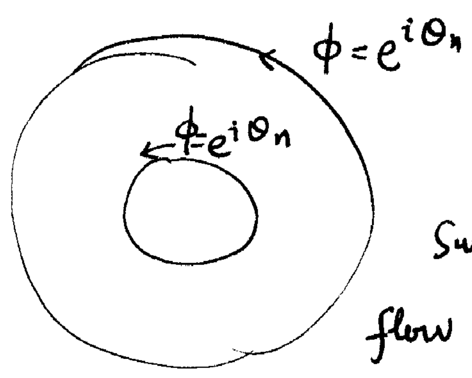
Let us twist φ_0 (boost)

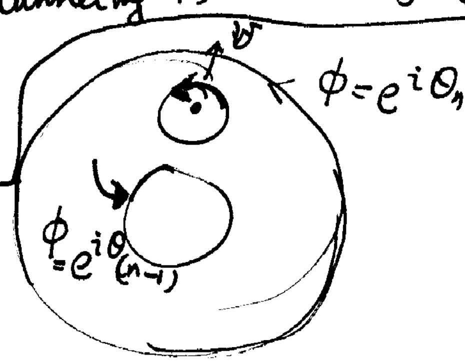


$\varphi' = \varphi_0\, e^{imvx}$ such that $mvL = 2\pi n$.

Because $|\varphi_0|$ is fixed, we cannot untwist the condensate without suppressing $|\varphi_0|$. to zero. Thus although $\varphi_0 e^{imvx}$ has a high energy, it cannot relax unless vortex tunneling.

The vortex tunneling is the decaying machanism of superfluidity.





Superfluid won't flow forever, but it is long-lived!

creat a vortex in the inner wall and move it accross the sample to outer wall untwist the superfluid.