Lect 8. Superfluid at finite temperatures, low dimensions

- we will use imaginary path integral

$$Z = \int D^2\varphi \ \exp\left[-\int_0^\beta d^dx\, d\tau \ \left[\frac{1}{2}(\varphi^*\partial_\tau\varphi - \varphi\partial_\tau\varphi^*) + \frac{1}{2m}\partial_x\varphi^*\partial_x\varphi - \mu|\varphi|^2 + \frac{V_0}{2}|\varphi|^4\right]\right.$$

in the time domain, we have a finite size $\beta = \frac{1}{k_B T}$. thus if the correlation length in the time domain ξ_T is much larger than β, then we can neglect the fluctuation in the τ-domain. (The relation between $\xi_T \sim \xi_0^z$, where z is called dynamic critical exponent.) In this case, we arrive at the ~~tat~~ classic partition function

- $$Z = \int D^2\varphi \ \exp\left[-\beta\int d^dx \ \left[\frac{|\varphi_0|^2}{2m}(\partial_x\theta)^2\right]\right\}$$ where only phase fluctuations are kept.

$\rightarrow \ S_{eff} = \int d^dx \ \frac{\eta}{2}(\partial_x\theta)^2, \quad \eta = \frac{|\varphi_0|^2}{m T k_B}, \quad T\eta$ is the phase rigidity.

Another important question is whether long range order can survive under thermal fluctuations.

Using the result in the last lecture, we have at $d \geq 3$, thermal fluctuations does not always destroys long range order.

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For $d<2$, thermal fluctuations always destroy superfluidity:

$$Z[J] = \int D\theta \; e^{-\int d^dx \, S \; + i\int d^dx \; J(x)\,\theta(x)} \qquad \text{where } J(x) = \delta(x) - \delta(x_0)$$

$$\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle = \frac{Z[J]}{Z[0]} = \frac{1}{Z(0)} \int D\theta \; \exp\Big[-\tfrac{1}{2}\int d^dx \, d^dx' \; \theta(x)\, G^{-1}(x,x')\,\theta(x') + iJ(x)\,\theta(x)\Big]$$

$$= \exp\Big[-\tfrac{1}{2}\int dx\, dx' \; J(x)\, G(x,x')\, J(x')\Big], \quad \text{where } G \text{ is the inverse of } -\partial_x^2$$

$\Rightarrow$.

$$\langle \theta(x_1)\,\theta(x_2)\rangle = \int D\theta \;\theta(x_1)\theta(x_2)\; e^{-\frac{1}{2}\int d^dx \;\theta(x)\, G^{-1}(x\,x')\,\theta(x')} \Big/ \int D\theta\, e^{-\int d\cdots}$$

$$= G(x_1, x_2)$$

$$\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle = e^{-[G(0,0) - G(x,0)]} = e^{\langle \theta(x)\theta(0)\rangle - \langle\theta(0)\theta(0)\rangle}$$

$$\langle \theta(x)\theta(0) - \theta(0)\theta(0)\rangle = -\int \frac{d^dk}{(2\pi)^d}\, \frac{1 - e^{ikx}}{\eta\, k^2}$$

$$= \begin{cases} -\dfrac{1}{\eta}\left[\dfrac{K_d\,\Lambda^{d-2}}{(d-2)}\right] & \text{for } x\to\infty \ (d>2) \quad \text{where } \Lambda = 1/\ell \\[2ex] \dfrac{-1}{\eta}\,\dfrac{1}{2\pi}\ln\left[\dfrac{|x|}{\Lambda^{-1}}\right] & \text{for } x \to \infty\ (d=2) \text{ — the wave-vector cut off} \\[2ex] \dfrac{-1}{\eta}\,\dfrac{1}{2}|x| & \text{for } x\to\infty\ (d=1) \end{cases}$$

at $d=1 \Rightarrow \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim e^{-\frac{1}{2\eta}|x|}$, which is exponentially decaying, with decay length $\underline{2\eta}$.

at $d=2 \Rightarrow \langle e^{i\theta(x)} e^{-i\theta(0)} \rangle \sim e^{-\frac{1}{2\pi\eta}\ln\frac{|x|}{\Lambda^{-1}}} = \left(\frac{\Lambda^{-1}}{|x|}\right)^{\frac{1}{2\pi\eta}}$

which has power-law decaying, with exponent $\frac{1}{2\pi\eta}$

for $d>2$, there's no infra-red divergence.

§2. K-T transition.

So far we have completely neglected the compactness of θ. we will see at 2D, it gives rise to the topological excitation of vortices, and its effect to change the transition to K-T type.

The action of a single vortex: $\varphi(x,y) = f(r)\, e^{i\phi}$, $f(r) = \begin{cases} 0 & r\to 0 \\ \varphi_0 & r\to\infty \end{cases}$

$$S_v = \int d^2r \ \frac{1}{2}\eta \frac{1}{r^2} + S_c = \eta\pi \ln\frac{L}{l} + S_c \leftarrow \text{core energy}$$

The interaction between two vortices with the same vorticity can be calculated by the analogy with 2D electro-statics.

$$\frac{1}{2}\eta(\nabla\theta)^2 \leftrightarrow \frac{E^2}{8\pi} \quad\Rightarrow\quad E = \sqrt{4\pi\eta}\ \nabla\theta = \sqrt{4\pi\eta}\ \frac{2\pi}{2\pi r} \qquad q = \frac{2\pi r}{4\pi}E = \frac{2\pi\cdot\sqrt{4\pi\eta}}{4\pi} = \sqrt{\pi\eta}$$

$$\Rightarrow \Delta E = \sqrt{4\pi\eta}\int_l^r dr\,\frac{1}{r}\cdot\sqrt{\pi\eta} = \underline{2\pi\eta\ln\frac{r}{l}}$$

To calculate the partition function, we need to include the vortex configuration. For a fixed vortex configuration $\varphi_c = e^{i\theta_c}$ and other fluctuations contribute as

$$Z = \int D\delta\theta \; e^{-\int d^2x \; \frac{\eta}{2}(\partial_x(\theta_c + \delta\theta))^2} = \int D\delta\theta \; e^{-\int d^2x \; \frac{\eta}{2}(\partial_x\theta_c)^2 + \frac{\eta}{2}(\partial_x\delta\theta)^2 + \eta(\partial_x\theta_c)(\partial_x\delta\theta)}$$

$$= e^{-S_{eff}(\theta_c)} \cdot \int D\delta\theta \; e^{-\int dx \frac{\eta}{2}(\partial_x\delta\theta)^2 + \int dx \; \eta \; \partial_x^2\theta_c \cdot \delta\theta} \approx 0$$

$$= e^{-S_{eff}(\theta_c)} \cdot Z_0 \leftarrow$$ contribution from vortex free configuration

($\because$ the vortex current is divergence free. $\nabla \cdot (j) = \partial^2\theta_c = 0$)

$\uparrow$ given by the long-range interaction between vortices.

$$Z = Z_0 \sum_n \frac{1}{n!\,n!} \int \prod_{i=1}^{2n} d^2r_c \; e^{-2nS_c} \cdot e^{\sum_{i<j}^{2n} (2\eta\pi)\, q_i q_j \ln \frac{r_{ij}}{\ell}}$$

we must have equal number of vortices and anti-vortices, to avoid energy divergence. $q_{i,j} = \pm 1$, satisfying $\sum q_i = 0$.

Let us estimate the effective action $Z = e^{-S_{eff}}$.

$$S_{eff} \sim \underset{\substack{\nearrow \\ \text{from } \frac{1}{n!n!}}}{2n \ln n} + n\Big(\underset{\uparrow}{2\eta\pi \ln \frac{\ell_n}{\ell}} + 2S_c\Big) - \underset{\substack{\uparrow \\ \text{integration area}}}{2n \ln \frac{L^2}{\ell^2}}$$

$$= L^2 \cdot \frac{2}{\ell_n^2}\left[(\eta\pi - 2)\ln\frac{\ell_n}{\ell} + S_c\right]$$

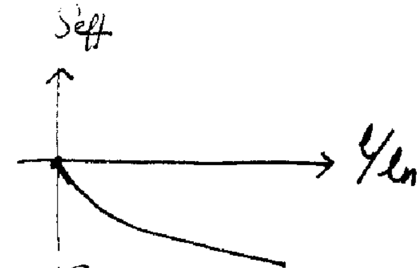
where ℓ_n is the average inter-vortex distance.

let us plot S_{eff} v.s ~~to~~ l/l_n.

If $S_c << -1$, thus it's cheep to creat vortices, we minimize S_{eff}

$$S_{eff} = 2\left(\frac{L}{l}\right)^2 \cdot \left(\frac{l}{l_n}\right)^{+2}\left[S_c - (\eta\pi-2)\ln\frac{l}{l_n}\right]$$

if $\eta\pi < 2$,



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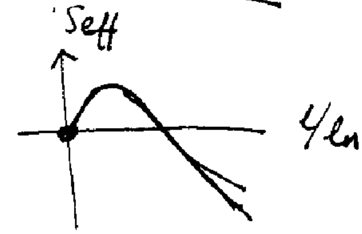
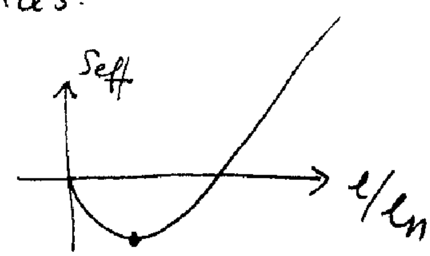


$\Rightarrow$ proliferation of vortices. $l_n \to l$

$\langle e^{i\theta(x)} e^{-i\theta(0)}\rangle \sim e^{-|x|/\xi}$, where $\xi \to l$.

if $S_c >> 1$, it's expensive to make vortices.

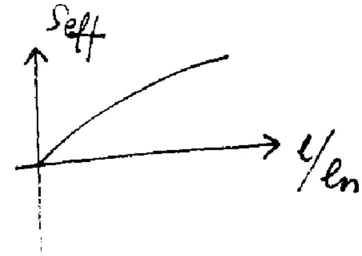


if $\eta\pi < 2$, high ~~temperature~~

the optimal $l/l_n \sim e^{-S_c/(2-\eta\pi)}$

i.e. $l_n \sim e^{\frac{S_c}{2-\eta\pi}} \cdot l$, a finite number of vortices.

if $\eta\pi > 2$. low lemperture $T < \frac{\pi}{2}\rho_s$



no free vortices.

Thus when vortices are expensive ($S_c \gg 1$), ~~we~~ we have a transition as increasing temperature.

$\xi \to +\infty$

$\eta_c = \frac{2}{\pi}$

short - range $\qquad$ quasi - long range

$$\boxed{\xi \sim l_n \sim l\, e^{\frac{S_c}{2-\eta_c \pi}}} \qquad \text{and} \qquad \eta = \frac{\rho_s}{T} \sim \frac{2}{\pi}$$

$$T_K = \frac{\pi}{2}\rho_s .$$

naively, we would expect

$$\xi \sim l\, e^{\frac{S_c}{2-\rho_s \pi / T}} \sim l\, e^{\frac{S_c T_K/2}{T - \frac{\pi}{2}\rho_s}} \propto l\, e^{\frac{E_s}{T-T_K}} .$$

but as $T \to T_K$, ρ_s changes. actually, $2-\eta_c^* \pi \propto (T-T_{TK})^{1/2}$.

and $\xi(T) \sim l\, e^{\left(\frac{E_s}{T-T_{KT}}\right)^{1/2}}$ (we will analysis it later).

§ Renormalization group & Scaling dimensions

Relevant perterbation can change the long distance behavior of the system. How to decide a perterbation is relevant or not can be learned from the scaling dimension analysis.

Say, at $e^{-S_c} \ll 1$, i.e. it seems that vortex is costly. But we know at $\eta\pi < 2$, no matter how small e^{-S_c} is, vortex will destroy the algebric correlation of $\langle e^{i\theta(x)} e^{i\theta(y)} \rangle$. Then it's a relevant perturbation. On the other hand, at $\eta\pi > 2$, vortex is an irrelevant perturbation.

Let us consider a theory of $S = S_0 + \int d^d\left(\frac{x}{\ell}\right) g\, O(x)$, in which

$$\langle O(x)\, O(y) \rangle \sim \frac{1}{\left|\frac{x-y}{\ell}\right|^{2\Delta}}$$, where ℓ is the short ~~energy~~ length scale.

At second order perturbation

$$Z = \int D\varphi\; e^{-S_0 + g\int d^d\left(\frac{x}{\ell}\right) O(x)} = \int D\varphi\; e^{-S_0} \left(1 + g\int d^d\left(\frac{x}{\ell}\right) O(x) + \frac{g^2}{2}\int d\frac{x}{\ell}\, d\frac{y}{\ell}\, \langle O(x)\, O(y)\rangle \ldots\right)$$

if $\langle O(x) \rangle = 0$

$$= \int D\varphi\; e^{-S_0}\; e^{\frac{g^2}{2}\int \frac{dx}{\ell}\frac{dy}{\ell} \langle O(x)\, O(y)\rangle}$$

$\Rightarrow$

$$\Delta S_{eff} = -\ln Z + \ln Z_0 \overset{\smile}{=} \ln\left[\frac{g^2}{2} \int \frac{d^d x}{\ell^d} \int \frac{d^d y}{\ell^d} \langle O(x)\, O(y) \rangle\right]$$

$$= -2\ln g - \ln \int \frac{d^d R}{\ell^d} \cdot \int \frac{d^d r}{\ell^d} \frac{1}{\left(r/\ell\right)^{2\Delta}}$$

$$= -2\ln g - \left[\ln\left(\frac{L}{\ell}\right)^d + \ln\left(\frac{L}{\ell}\right)^{d-2\Delta}\right] = -2\ln g - 2(d-\Delta)\ln\frac{L}{\ell}$$

if $\Delta < d$ and $L > \xi = g^{\frac{-1}{d-\Delta}}$ $\Rightarrow$ $\Delta S_{eff} < 0$, the system prefers to have two "O"-operators appear at short-distance (at the order of ξ). Thus if we are interested at long distance behavior, perturbation of O' is always important. On the other hand if $\Delta > d$, it's irrelavant for long range correlations.

§ The duality between 2D xy-model and the 2D clock model

Let us conside a generic clock model

$$S = \int d^2x \left[\frac{K}{2} (\partial_x \varphi)^2 - g \cos(n\varphi) \right] \qquad Z_n\text{- symmetry} \quad \varphi \to \varphi + \frac{2\pi}{n}.$$

here φ is non-compact.

say, at $n=1$.

$$Z = \int D\varphi \, e^{-\int d^2x \frac{K}{2} (\partial_x \varphi)^2} \, e^{-g\int \cos\varphi \, dx}$$

$$= \int D\varphi \, e^{-\int d^2x \frac{K}{2} (\partial_x \varphi)^2} \cdot \sum_k \frac{g^k}{k!} \left[\int d^2x \, \frac{e^{i\theta} + e^{-i\theta}}{2} \right]^k$$

$$= Z_0 \sum_k \frac{1}{k!k!} \int \prod_{i=1}^{2k} \left(\frac{g}{2}\right)^{2k} \langle e^{i\theta(r_1)} e^{i\theta(r_2)} \cdots e^{i\theta(r_k)} \, e^{-i\theta(r_{k+1})} \cdots e^{-i\theta(r_{2k})} \rangle$$

how to calculate $\left\langle e^{i\left(\sum_{i=1}^{k} \theta(r_i) - \sum_{i=k+1}^{2k} \theta(r_i)\right)} \right\rangle$

Again, we introduce $\underbrace{\text{field}}_{\text{source}}$ $J(x) = \sum_{i=1}^{k} \delta(x-r_i) - \sum_{i=k+1}^{2k} \delta(x-r_i)$

$$\left\langle e^{i\left(\sum_{i=1}^{k} \theta(r_i) - \sum_{i=k+1}^{2k} \theta(r_i)\right)} \right\rangle = \exp\left[-\frac{1}{2}\int dx\,dx'\, J(x)\, G(x-x')\, J(x')\right],$$

where G is the inverse of $-K\partial_x^2$

$$= \exp\left[-\sum_{i<j} q_i q_j \langle \theta(r_i)\theta(r_j)\rangle - \frac{1}{2}\sum_{i=1}^{2K} q_i^2 \langle \theta(r_i)\theta(r_i)\rangle\right]$$

$$= \exp\left[+\sum_{i<j} q_i q_j \frac{1}{2\pi K} \ln\frac{|r_i-r_j|}{L}\right] \cdot \exp\left[+\frac{1}{2}\sum_{i=1}^{2K} q_i^2 \ln\frac{l}{L}\right]_{\times\frac{1}{2\pi K}}$$

$$= \exp\left[+\sum_{i<j} q_i q_j \frac{1}{2\pi K} \ln\frac{|r_i-r_j|}{l}\right] \qquad \exp\left[-\underbrace{\frac{1}{2}}_{\frac{1}{2\pi K}}\left(\sum_{i=1}^{2K} q_i\right)^2 \ln\frac{l}{L}\right]$$

$$= \left[\frac{|r_i-r_j|}{l}\right]^{+\frac{q_i q_j}{2\pi K}}$$

$$\Rightarrow Z = Z_0 \sum_k \frac{1}{k!\,k!}\int \prod_{i=1}^{2K} e^{-2KS_c} \cdot e^{\sum_{i<j}^{2K} \frac{q_i q_j}{2\pi K} \ln\frac{r_{ij}}{l}}$$

which is the same as vortex partition function, if

$$\boxed{e^{-S_c} = g/2, \quad \text{and} \quad 2\eta\pi = \frac{1}{2\pi K}.}$$

Then the vortex in the xy model (non-local object), maps into $e^{i\varphi(x)}$ operator (local) in the colock model (dual representation).

xy model $\xleftrightarrow{\text{dual to}}$ Clock model

$$S = \int \frac{\eta}{2} (\partial_x \theta)^2 \, dx^2 \qquad \boxed{\begin{array}{c} 2\eta\pi = \frac{1}{2\pi K} \\ e^{-S_c} = g/2 \end{array}} \qquad S' = \int dx^2 \, \frac{K}{2} (\partial_x \varphi)^2 + g \cos\varphi$$

compact, allow vortex $(\theta + 2\pi = \theta)$

non-compact, no vortex $(\varphi + 2\pi \neq \varphi)$

U(1) symmetry, no vertex operator

vertex operator $e^{\pm i\varphi}$

non-local excitation : vortex $\longleftrightarrow$ vertex operator (local operator)

how about compact clock model

φ is also compact

$$S = \int \frac{\eta}{2} (\partial_x \theta) + g \cos\theta \quad \longleftrightarrow \quad S = \int \frac{K}{2} (\partial_x \varphi) + g \cos\varphi$$

vortex of the $\underbrace{\text{field}}_{\text{dual}}$ φ.

vortex fluctuation in the original xy model

Lect 9: RG analysis to K-T transition

Let us consider the dual theory of XY model: the non-compact clock model
$$S = \int d^2x \ \frac{K}{2} (\partial_x \varphi)^2 + g \cos n\varphi$$
where we set "n" to a general number.

as we know $\langle e^{in\theta(x)} e^{-in\theta(0)} \rangle = \left(\frac{\ell}{|x|}\right)^{\frac{n^2}{2\pi K}} \Rightarrow$ the value of the correlation function depends on the short length scale! In order to have a well-defined field theory, we must explicitly specify a short length scale! i.e. we would like to write

$$S = \int d^2x \ \frac{K_\ell}{2} (\partial_x \theta_\ell)^2 - g_\ell \cos\theta_\ell, \quad \text{where } \theta_\ell(x) = \int_{|k|<\frac{2\pi}{\ell}} d^2k \ \theta_k e^{i\vec{k}\vec{x}}$$

we have three parameters K_ℓ, g_ℓ and "ℓ", and we can make two dimensionless parameters K_ℓ, and $\bar{g}_\ell = g_\ell \ell^2$.

We would think there are two different phases: ~~At~~ small ~~large~~ K and small $\bar{g}_\ell$, fluctuations are large and pinning potentials are weak, we have a Z_n-symmetric state. On the other hand, if K_ℓ ~~are~~ $\bar{g}_\ell$ large, pinning potentials are ~~is~~ strong and fluctuations are weak, we have a symmetry breaking ground state.

Let us take $\cos n\varphi$ term as perturbation. As we explained before the scaling dimension $\Delta = \frac{n^2}{4\pi \chi_\ell}$, thus at $\frac{n^2}{4\pi\chi_\ell} > 2$ (i.e. $\chi_\ell < \frac{n^2}{8\pi}$), the clock term is irrelevant; at $\frac{n^2}{4\pi\chi_\ell} < 2$ (i.e. $\chi_\ell > \frac{n^2}{8\pi}$), the clock term is relevant. Thus $\chi_\ell \gtrless \frac{n^2}{8\pi}$, we suppose to have two different phases. We'll use RG to confirm it.

let us change the short range cut off $\ell \to \ell' = \ell + \Delta\ell$, and we will see how these coupling constant changes. We seperate the fast and slow modes

$\theta_\ell = \theta_{\ell'} + \delta\theta$, where $\delta\theta$ is the fast mode, containing modes with wavelengths between ℓ and $\ell + \Delta\ell$.

$$(\partial_x \theta_\ell)^2 = (\partial_x \theta_{\ell'})^2 + (\partial_x \delta\theta)^2 \quad \text{(no mixing after integration } \int d^2x)$$

$$\cos n\theta_\ell = \cos(n\theta_{\ell'} + n\delta\theta) = \cos n\theta_{\ell'} - n \sin n\theta_{\ell'}\, \delta\theta - \frac{n^2}{2}\cos n\theta_{\ell'} (\delta\theta)^2$$

$$\Rightarrow S = \int d^2x \; \frac{\chi_\ell}{2}(\partial_x\theta_{\ell'})^2 - g_\ell \cos n\theta_{\ell'} + \frac{\chi_\ell}{2}(\partial_x \delta\theta)^2$$

$$\int d^2x \; + n g_\ell \sin n\theta_{\ell'}\, \delta\theta + \frac{n^2}{2} g_\ell \cos n\theta_{\ell'} (\delta\theta)^2$$

we will treat $\theta_{\ell'}$ as background field and integrate out the fast field of $\delta\theta$.

This will generate a number of terms such as $(\partial_x \theta_{\ell'})^2$, $\cos 2n\theta_{\ell'} (\partial_x\theta_{\ell'})^2$, $\cos 2n\theta_{\ell'}$, $\cdots$, but we only look at the term that we've already had.

$$+\int d^2x\, d^2y\, \frac{n^2 g_\ell^2}{8} (\partial_x\theta_{\ell'})^2 \langle n\delta\theta(x)\, n\delta\theta(y)\rangle \cdot (\vec{x}-\vec{y})^2$$

$$\Rightarrow \quad \chi_{\ell+\delta\ell} = \chi_\ell + \frac{n^2}{8} g_\ell^2 K_2, \quad \text{where } K_2 = \int d^2x\, |x|^2 \cdot \langle n\delta\theta(x)\, n\delta\theta(0)\rangle$$

$$K_2 = \int_{\frac{2\pi}{\ell+\delta\ell}<|k|<\frac{2\pi}{\ell}} \frac{d^2k}{(2\pi)^2} \int d^2x\, \frac{n^2 x^2}{\chi_\ell k^2}\, e^{ikx\cdot\cos\theta - 0^+|x|}$$

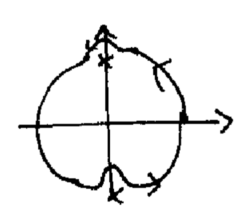
$$= \int d^2x\, \frac{n^2}{\chi_\ell} \cdot x^2 \int_0^{2\pi} d\theta\, e^{i\cdot\frac{2\pi}{\ell}x\cos\theta - 0^+ x} \cdot \frac{1}{(2\pi)^2} \ln\left(\frac{\ell+\delta\ell}{\ell}\right)$$

$$= \frac{n^2}{\chi_\ell}\frac{1}{4\pi^2}\frac{\Delta\ell}{\ell}\cdot\int_0^{2\pi} d\theta \int x^3 dx\, e^{i\frac{2\pi}{\ell}(\cos\theta + i0^+)x} \cdot 2\pi$$

$$= \frac{n^2}{\chi_\ell}\cdot\frac{1}{2\pi}\frac{\Delta\ell}{\ell}\int_0^{2\pi} d\theta \cdot \frac{1}{\left(\frac{2\pi}{\ell}\right)^4(\cos\theta+i0^+)^4}\int_0^{+\infty} x'^3\, dx'\, e^{ix'(1+i0^+)}$$

$$= \frac{\Delta\ell}{\ell}\frac{n^2}{\chi_\ell}\frac{\ell^4}{32\pi^5}\int_0^{2\pi} d\theta\, \frac{1}{(\cos\theta+i0^+)^4}\cdot 6 = \frac{\Delta\ell}{\ell}\frac{3n^2\ell^4}{16\pi^5}\int d\theta\, \frac{1}{(\cos\theta+i0^+)^4}$$

$$= \frac{\Delta\ell}{\ell}\frac{3n^2\ell^4}{2\pi^4\chi_\ell}$$



$$\cos\theta \to \frac{z+z^{-1}}{2}$$

$$\Rightarrow \boxed{\frac{d\chi_\ell}{d\ln\ell} = \frac{3n^4(g_\ell a^2)^2}{16\pi^4\chi_\ell}}$$

$$Z = \int D\theta_{\ell'}\, D\delta\theta \; e^{-\int d^2x \left[\frac{K_\ell}{2}(\partial_x\theta_{\ell'})^2 - g_\ell \cos\theta_{\ell'}\right]} \cdot e^{-\int d^2x \frac{K_\ell}{2}(\partial_x\delta\theta)^2}$$

$$\cdot \exp\left(\int d^2x \; - n g_\ell \sin n\theta_{\ell'}\, \delta\theta - \frac{n^2}{2} g_\ell \cos n\theta_{\ell'} (\delta\theta)^2 \right)$$

$$= \int D\theta_{\ell'}\, e^{-\int d^2x \left(\frac{K_\ell}{2}(\partial_x\theta_{\ell'})^2 - g_\ell\cos\theta_{\ell'}\right)} \; e^{-\int d^2x \frac{n^2}{2} g_\ell \cos n\theta_{\ell'} \langle(\delta\theta)^2\rangle + \int d^2x\, d^2y \frac{1}{2} g_\ell^2 \sin n\theta_\ell(x) \sin n\theta_\ell(y) \langle\delta\theta_\ell(x)\,\delta\theta(y)\rangle}$$

$$\Rightarrow \delta S = \int d^2x \; \frac{1}{2} g_\ell \cos n\theta_{\ell'} \underset{n^2}{\langle \delta\theta(0)\,\delta\theta(0)\rangle}$$

$$- \int d^2x\, d^2y \; \frac{1}{2}(g_\ell)^2 \sin n\theta_{\ell'}(x) \sin n\theta_{\ell'}(y) \; \langle n\delta\theta(x)\; n\delta\theta(y)\rangle .$$

$$\langle n^2 (\delta\theta(0))^2\rangle = \int_{\frac{2\pi}{\ell+\delta\ell} < k < \frac{2\pi}{\ell}} \frac{d^2k}{(2\pi)^2} \frac{n^2}{\chi_\ell |k|^2} = \frac{n^2}{2\pi\chi_\ell} \ln\left(\frac{\ell+\delta\ell}{\ell}\right)$$

Thus the first term will change $g_\ell \to g_{\ell'} = g_\ell - \frac{1}{2} g_\ell \frac{n^2}{2\pi\chi_\ell} \ln(1+\frac{\delta\ell}{\ell})$

i.e. $dg_\ell = - g_\ell \frac{n^2}{4\pi\chi_\ell} \cdot \frac{d\ell}{\ell} \Rightarrow \boxed{\frac{dg_\ell}{d\ln\ell} = - g_\ell \frac{n^2}{4\pi\chi_\ell}}$

$$\downarrow \quad \boxed{\frac{d(g_\ell \ell^2)}{d\ln\ell} = \left(2 - \frac{n^2}{4\pi\chi_\ell}\right)(g_\ell \ell^2)}$$

The second term can be represented as

$$\int d^2x\, dy^2 \; \frac{g_\ell^2}{4} \left[\sin n\theta_{\ell'}(x) - \sin n\theta_{\ell'}(y)\right]^2 \langle n\delta\theta(x)\, n\delta\theta(y)\rangle - \int d^2x\, dy^2 \; \frac{g_\ell^2}{2} \sin^2 n\theta(x) \langle n\delta(x)\, n\delta(y)\rangle$$

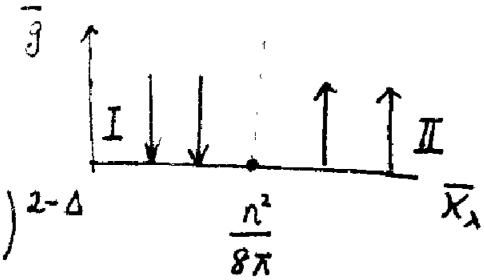
$$= \int d^2x\, dy^2 \; \frac{n^2 g_\ell^2}{4} \cos^2 n\theta_{\ell'} (\partial_x\theta_{\ell'})^2 (x-y)^2 \langle n\delta\theta(x)\; n\delta\theta(y)\rangle$$

$$- \int dx^2 \; \frac{g_\ell^2}{2} \sin^2 n\theta_{\ell'}(x) \int dy^2 \; \langle n\delta\theta(x)\, n\delta\theta(y)\rangle$$

§ RG flow, fixed point $\qquad \bar{g} = g\ell^2$, and $\bar{K} = K$, are dimensionless.

$$\frac{d\bar{g}}{d\ln \ell} = \left(2 - \frac{n^2}{4\pi \bar{K}}\right)\bar{g} \; ; \qquad \frac{d\bar{K}}{d\ln \ell} = \frac{3n^4\,\bar{g}^2}{16\pi^4\,\bar{K}}$$

☆ let us first ignore the second equation



$$\bar{g}(\ell) = \bar{g}(\ell_0)\, e^{\left(2 - \frac{n^2}{4\pi\bar{K}}\right)\ln\left(\frac{\ell}{\ell_0}\right)} = \bar{g}(\ell_0)\left(\frac{\ell}{\ell_0}\right)^{2-\Delta}$$

$$\Delta = \frac{n^2}{4\pi\bar{K}}$$

if $2-\Delta > 0$, no matter how small $\bar{g}(\ell_0)$ is, it can become as large as you want. Let us set the length scale of $\bar{g}(\ell) \sim 1 = \bar{g}(\ell_0)\left(\frac{\ell}{\ell_0}\right)^{2-\Delta}$

$\Rightarrow \quad \ell = \ell_0\left(\frac{1}{\bar{g}_{\ell_0}}\right)^{\frac{1}{2-\Delta}}$, at this scale RG fails, the perturbative method does not apply.

This implies we enter the length scale of correlation length $\xi \sim \ell_0\left(\frac{1}{\bar{g}_{\ell_0}}\right)^{\frac{1}{2-\Delta}}$.

(Symmetry breaking state) $\boxed{\langle e^{i\varphi}\rangle \neq 0, \rightarrow \text{vortex condensation.}}$

if $2-\Delta < 0$, then $\bar{g}$ dies exponetially, and we arrive at a symmetric phase.

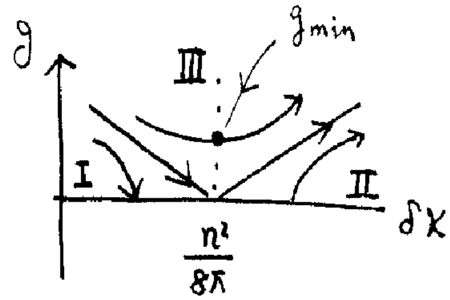
↖ powerlaw superfluid phase.

☆ Near the transition point at $K = \frac{n^2}{8\pi}$, the changes of $\bar{g}$ and $\bar{K}$ are comparable to each other. We need to conside both simultaneously.

Let us linearize the RG around $\bar{X} = \frac{n^2}{8\pi}$, $\bar{g} = 0$. $\Rightarrow$

$$\frac{d\bar{g}}{d\ln l} = \frac{16\pi}{n^2}\,\delta\bar{X}\,\bar{g}\,, \qquad \frac{d\delta\bar{X}}{d\ln l} = \frac{3n^2\bar{g}^2}{2\pi^3}$$

$$\Rightarrow \bar{g}\,d\bar{g} - \frac{3n^4}{32\pi^4}\,\delta\bar{X}\,d\delta\bar{X} = 0 \quad \text{i.e.} \quad (\delta\bar{X})^2 = \frac{3n^4}{32\pi^4}\,\bar{g}^2 + \text{Const}$$



In region I, we have the solution

$$\delta\bar{X}_l = -\sqrt{\frac{3n^4}{32\pi^4}\,\bar{g}^2 + (\delta X_\infty)^2}$$

In region II,

$$\delta\bar{X}_l = \sqrt{\frac{3n^4}{32\pi^4}\,\bar{g}^2 + (\delta X(l_0))^2}$$

In region III

$$\bar{g}_l = \sqrt{\frac{32\pi^4}{3n^4}(\delta\bar{X}_l)^2 + (\bar{g}_{min})^2}\,.$$

Regions I, II are approximately described by neglecting the renormalization of δX.

Regions III is the crossover region

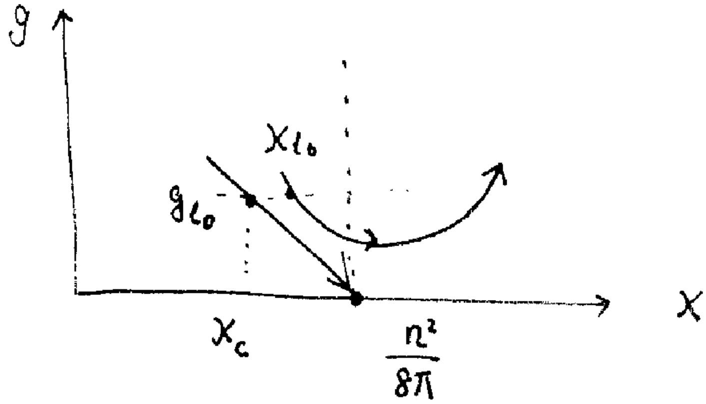
$$\frac{d\delta\bar{X}}{\frac{32\pi^4}{3n^4}(\delta\bar{X}_l)^2 + (\bar{g}_{min})^2} = n^2\,d\ln l$$

, we integrate out from l to ξ

$$\Rightarrow \int_{\delta X(l_0)}^{\delta X(\xi)} \frac{d\delta\bar{X}}{\frac{32\pi^4}{3n^4}(\delta\bar{X}_l)^2 + (\bar{g}_{min})^2} = n^2 \ln\left(\frac{\xi}{l}\right).$$

At ξ, $\bar{g}$ and δX are at the order of 1

$\Rightarrow$ relation between ξ and $\delta X(l_0)$



along the line of

$$\delta X_c = -\sqrt{\frac{3n^4}{32\pi^4}}\ \bar{g}_l ,$$

$$\Rightarrow g_{min} = 0 ,$$

The left hand side diverges as

$\delta X(\xi) \to 0$, thus $\xi \to +\infty$.

let set X_{l_0} slightly larger than X_c, ~~then $\bar{g}^2_{min} = 2\left(\frac{32\pi^4}{3n^4}\right)^{1/2}$~~

and fix g_{l_0}.

$$\bar{g}^2_{l_0} - \frac{32\pi^4}{3n^4}(\delta X_c)^2 = 0$$

$$\bar{g}^2_{l_0} - \frac{32\pi^4}{3n^4}\left[(\delta X_c)^2 + 2\delta X_c\,(X_{l_0} - X_c)\right] = g^2_{min}$$

$$\Rightarrow \bar{g}^2_{min} = 2\left(\frac{32\pi^4}{3n^4}\right)^{1/2} \bar{g}_{l_0}\,(X_{l_0} - X_c)$$

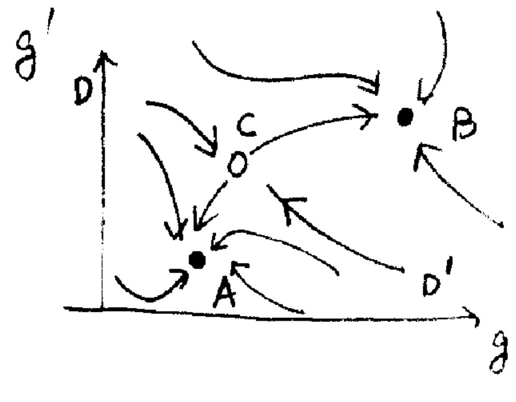
The integration can be performed from ($\delta X = 0$. $g = g_{min}$)

$$\Rightarrow \int_0^{\delta X(\xi)=1} \frac{d\,\delta X}{\frac{32\pi^4}{3n^4}(\delta X)^2 + (g_{min})^2} \simeq \int_{\sqrt{\frac{3n^4}{32\pi^4}}\,g_{min}}^{\delta X(\xi)=1} \frac{d\,\delta X}{\frac{32\pi^4}{3n^4}(\delta X)^2} = n^2 \ln\left(\frac{\xi}{l_0}\right)$$

$$-\frac{3n^4}{32\pi^4}\frac{1}{\delta X}\Bigg|_{\sqrt{\frac{3n^4}{32\pi^4}}\,g_{min}}^{1} \simeq \sqrt{\frac{3}{32}}\ \frac{n^2}{\pi^2}\ \frac{1}{g_{min}} = n^2 \ln\left(\frac{\xi}{l_0}\right)$$

$$\Rightarrow \xi \simeq l_0\, e^{\left[\frac{C}{X_l - X_c}\right]^{1/2}} \qquad C = \frac{n^2}{2\pi\,\bar{g}_{l_0}}\left(\frac{3}{32\pi^2}\right)^{3/2}$$

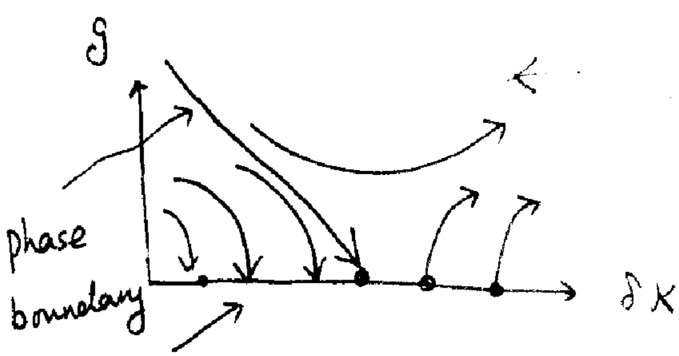
general principle of RG. Fixed points / phase transitions



Stable fixed points. A, B corresponds to stable phases.

unstable fixed points controls phase transitions C. The line $D \to C \to D'$ is the phase boundary between phase A/B.

K-T transition.



← symmetry breaking phase in the clock model.

(vortex condensation / disordered phase of the XY model).

a line of stable fixed point.

low-~~temperature~~, superfluid phase.

power law - correlation