Lect 14: Superconductivity — mean-field theory

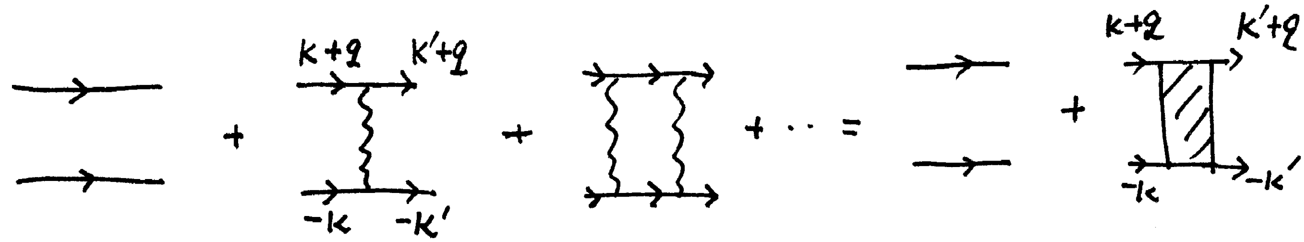
§ BCS Hamiltonian:

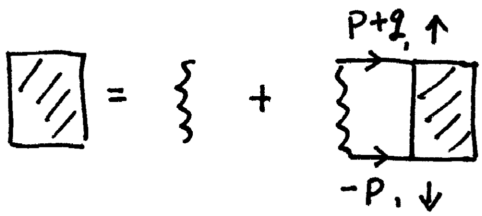
$$H = \sum_{k\sigma} \varepsilon_k C^+_{k\sigma} C_{k\sigma} - \frac{g}{V} \sum_{kk'q} C^+_{k+q\uparrow} C^+_{-k\downarrow} C_{-k'+q\downarrow} C_{k'\uparrow} ,$$

consider the vertex (p-p channel)

$$C(q,\tau) = \frac{1}{V^2} \sum_{kk'} \langle \bar\psi_{k+q\uparrow}(\tau) \bar\psi_{-k\downarrow}(\tau) \psi_{-k'\downarrow}(0) \psi_{k'+q,\uparrow}(0) \rangle$$

C =





$$\Rightarrow \Gamma_q = g + \frac{g}{V} \frac{1}{\beta} \sum_p G_{p+q} G_{-p} \Gamma_q$$

$$\Rightarrow \Gamma_q = \frac{g}{1 - \frac{g}{V} \frac{1}{\beta} \sum_p G_{p+q} G_{-p}}$$

$$\frac{1}{\beta} \sum_{ip_n} G_{p+q} G_{-p} = \frac{1}{\beta} \sum_{ip_n} \frac{1}{iq_n + ip_n - \xi_{p+q}} \frac{1}{-ip_n - \xi_p}$$

$$= \frac{1}{\beta} \frac{1 - n_f(\xi_{p+q}) - n_f(\xi_{-p})}{iq_n + \xi_{p+q} + \xi_{-p}}$$

$$\frac{1}{\beta V} \sum_p G_p G_{-p} = \int_{-\omega_D}^{\omega_D} d\epsilon\, \nu(\epsilon) \frac{1 - 2n_F(\epsilon)}{2\epsilon} \simeq \nu \int_T^{\omega_D} \frac{d\epsilon}{\epsilon} = \nu \ln\left(\frac{\omega_D}{T}\right)$$

↑
set $q = (0,0)$

$$\Rightarrow \boxed{\Gamma_{(0,0)} \simeq \frac{g}{1 - g\nu \ln \frac{\omega_D}{T}}}$$

divergence at $T_c \sim \omega_D e^{-\frac{1}{|g|\nu}}$.

§ mean field theory & pseudo-spin picture : ← generally g should be $g(k,k')$

- Define $\Delta_k = -\frac{g(kk')}{V}\sum_k \langle | c_{-k\downarrow} c_{k\uparrow} | \rangle$, $\Delta_k^* = -\frac{g(kk')}{V}\sum_k \langle | c^+_{k\uparrow} c^+_{-k\downarrow} | \rangle$,

$$\Rightarrow H = \sum_k \left[\xi_k c^+_{k\sigma} c_{k\sigma} + (\Delta^*_k c_{-k\downarrow} c_{k\uparrow} + \Delta_k c^+_{k\uparrow} c^+_{-k\downarrow})\right] + \frac{1}{V}\sum_{k} \Delta(k) g^{-1}(k,k')\Delta(k')$$

$$= \sum_k \left(c^+_{k\uparrow} \quad c_{-k\downarrow}\right) \begin{pmatrix} \xi_k & \Delta_k \\ \Delta^*_k & -\xi_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix} + \sum_k \xi_k + \frac{1}{V}\sum_k \Delta(k) g^{-1}(k,k')\Delta(k')$$

define Bogolibov transformation

$$\begin{pmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} c_{k\uparrow} \\ c^+_{-k\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix} \begin{pmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{pmatrix}$$

$$\Rightarrow H = \sum_k (\alpha^+_{k\uparrow}, \beta_{-k\downarrow}) \begin{bmatrix} \cos\theta_k & \sin\theta_k \\ -\sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} \xi_k & \Delta_k \\ \Delta^*_k & -\xi_k \end{bmatrix} \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \begin{bmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{bmatrix}$$

$$= \sum_k (\alpha^+_{k\uparrow}, \beta_{-k\downarrow}) \begin{bmatrix} \xi_k \cos 2\theta_k + \Delta_k \sin 2\theta_k, & -\xi_k \sin 2\theta_k + \Delta_k \cos 2\theta_k \\ -\xi_k \sin 2\theta_k + \Delta_k \cos 2\theta_k, & -\xi_k \cos 2\theta_k - \Delta_k \sin 2\theta_k \end{bmatrix} \begin{bmatrix} \alpha_{k\uparrow} \\ \beta^+_{-k\downarrow} \end{bmatrix}$$

Set $\tan 2\theta_k = \frac{\Delta_k}{\xi_k}$ & $\cos 2\theta_k = \frac{\xi_k}{E_k}$, $\sin 2\theta_k = \frac{\Delta_k}{E_k}$, where $E_k = \sqrt{\xi_k^2 + \Delta_k^2}$

$$\Rightarrow H = \sum_k (\alpha^+_{k\uparrow}\alpha_{k\uparrow} - 1/2) E_k + (\beta^+_k \beta_k - 1/2) E_k$$

$$\cos^2\theta_k = \frac{1}{2}\left(1 + \frac{\xi_k}{E_k}\right), \quad \sin^2\theta_k = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right).$$

- The ground state is the vaccum of Bogolinbov quasiparticle

$$|\Omega_s\rangle = \prod_k \alpha_{k\uparrow}\beta_{-k\downarrow} \Big| \underset{\text{of } c\, c^+}{\text{Vaccume}} \Big\rangle = \prod_k \left(-\sin\theta_k \cos\theta_k c_{k\uparrow} c^+_{k\uparrow} - \sin^2\theta_k c^+_{-k\downarrow} c^+_{k\uparrow}\right) |\text{vac}\rangle$$

$$\sim \prod_k \left[\cos\theta_k + \sin\theta_k\, c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow}\right] |vac\rangle$$

What's the meaning of $\alpha^\dagger_{k\uparrow}|\Omega_s\rangle$, $\beta^\dagger_{k\uparrow}|\Omega_s\rangle$, $\alpha^\dagger_{k\uparrow}\beta^\dagger_{k\uparrow}|\Omega_s\rangle$?

$$\alpha^\dagger_{k\uparrow}|\Omega_s\rangle = \left[\cos\theta_k\, c^\dagger_{k\uparrow} + \sin\theta_k\, c_{-k\downarrow}\right]\left[\cos\theta_k + \sin\theta_k\, c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow}\right]\left[\prod_{k'} \cdots\right]|vac\rangle$$

$$= \left[\cos^2\theta_k\, c^\dagger_{k\uparrow} + \sin^2\theta_k\, c^\dagger_{k\uparrow}\right]\left[\prod_{k'} (\cos\theta_k + \sin\theta_k\, c^\dagger_{-k\downarrow} c^\dagger_{k\uparrow})\right]|vac\rangle$$

$$= c^\dagger_{k\uparrow}\left[\prod_{k'} \cdots\right]|vac\rangle$$

$$\Rightarrow \beta^\dagger_{-k\downarrow}|\Omega_s\rangle = \underbrace{c^\dagger_{-k\downarrow}}_{[\prod_{k'}\cdots]}|vac\rangle \quad \& \quad \underline{\alpha^\dagger_{k\uparrow}\beta^\dagger_{-k\downarrow}|\Omega_s\rangle = (\sin\theta_k - \cos\theta_k\, c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})\left[\prod_{k'}\cdots\right]|vac\rangle}.$$

excited pair state

gap equation:

$$\Delta_k = -\frac{1}{V}\sum_{k'} g(k,k')\langle c_{-k\downarrow} c_{k\uparrow}\rangle,$$

$$\langle c_{-k\downarrow} c_{k\uparrow}\rangle = +\sin\theta_k\cos\theta_k\left[\langle\alpha^\dagger_{k\uparrow}\alpha_{k\uparrow}\rangle - \langle\beta_{-k\downarrow}\beta^\dagger_{-k\downarrow}\rangle\right] = -\frac{1}{2}\sin 2\theta_k \tanh\frac{\beta E_k}{2}$$

$$\Rightarrow \boxed{\Delta_k = \int\frac{dk'}{(2\pi)^3}\, g(k,k')\, \frac{\Delta_{k'}/2}{\sqrt{\xi_{k'}^2+\Delta_{k'}^2}}\tanh\frac{\beta}{2}\sqrt{\xi_{k'}^2+\Delta_{k'}^2}} \leftarrow \text{BCS gap Eq.}$$

§ analysis of gap Eq. — Consider the simplest case $|g(k,k')| = g$.

$$\Delta = gN(0)\int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon\, \frac{\Delta}{2E_k}\tanh\frac{\beta E_k}{2} \Rightarrow \boxed{\frac{1}{gN(0)} = \int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon\, \frac{1}{2E_k}\tanh\frac{\beta E_k}{2}}$$

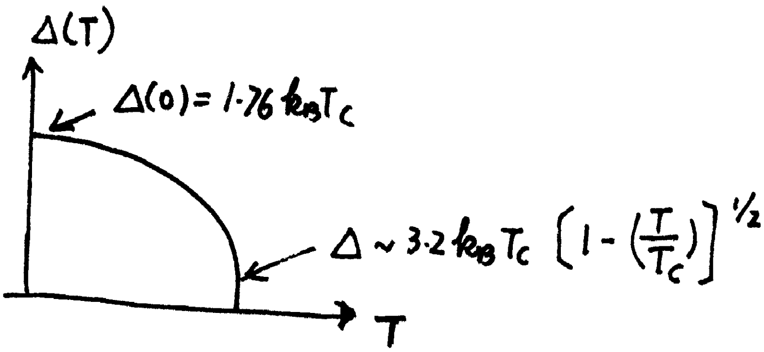
1° $T = 0\,K \Rightarrow \frac{1}{gN(0)} = \int_0^{\hbar\omega_D} d\epsilon \, \frac{1}{\sqrt{\epsilon^2+\Delta^2}} = \sinh^{-1} \frac{\hbar\omega_D}{\Delta}$

$$\Delta = \frac{\hbar\omega_D}{\sinh \frac{1}{gN(0)}} \sim 2\hbar\omega_D \, e^{\frac{-1}{gN(0)}} \quad \text{at } gN(0) \ll 1$$

2° $T \to T_c$ $\Delta \to 0$: $\frac{1}{gN(0)} = \int_0^{\hbar\omega_D} d\epsilon \, \frac{1}{\epsilon} \tanh\frac{\beta_c \epsilon}{2} = \ln(1.14\, \beta_c \hbar\omega_D)$

$$\Rightarrow k_B T_c = 1.14\,\hbar\omega_D \, e^{\frac{-1}{gN(0)}} \quad \Rightarrow \left(\frac{\Delta}{k_B T_c}\right)_{BCS} \approx 1.76 \quad \text{weak coupling SC}$$

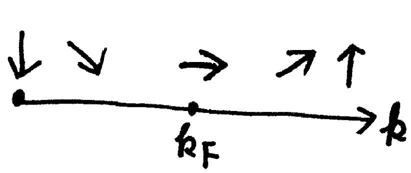
2.3 Hg
2.15 Pb
3-12 high Tc.



§ pseudo-spin picture (from Anderson)

$H(k) = \xi_k \tau_3 + \mathrm{Re}\Delta\, \tau_1 + \mathrm{Im}\Delta\, \tau_2$, we can take $\begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{pmatrix}$ as

Nambu spinor as pseudo spin-1/2. For each momentum k, there's a pseudo-magnetic field $\vec{B}(k) = (\mathrm{Re}\Delta, \mathrm{Im}\Delta, \xi_k)$.



← orientation of pseudo-spin.

SC: pseudo-spin spiral state in the k-space.

§ Green's function: $\psi(k) = \begin{pmatrix} c_{k\uparrow} \\ c^{+}_{-k\downarrow} \end{pmatrix}$

$$\mathcal{G}(\tau) = -T_\tau [\psi_k(\tau)\,\psi_k^{+}(0)] = -T_\tau \begin{bmatrix} c_{k\uparrow}(\tau)\, c^{+}_{k\uparrow}(0), & c_{k\uparrow}(\tau)\, c_{-k\downarrow}(0) \\ c^{+}_{-k\downarrow}(\tau)\, c^{+}_{k\uparrow}(0), & c^{+}_{-k\downarrow}(\tau)\, c_{-k\downarrow}(0) \end{bmatrix}$$

$$\Rightarrow \mathcal{G}(k, i\omega_n) = [i\omega_n - H(k)]^{-1} = [i\omega_n - \xi_k \tau_3 - \Delta \tau_1]^{-1}$$

$$= \frac{i\omega_n + \xi_k\tau_3 + \Delta\tau_1}{(i\omega_n)^2 - [\xi_k^2 + \Delta^2]} = \frac{1}{2}\left[\frac{1 + \frac{\xi_k\tau_3 + \Delta\tau_1}{E_k}}{i\omega_n - E_k} + \frac{1 - \frac{\xi_k\tau_3+\Delta\tau_1}{E_k}}{i\omega_n + E_k} \right].$$

DOS: $g(\epsilon) = \frac{1}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left(A_{\uparrow\uparrow}(k,\epsilon) + A_{\downarrow\downarrow}(k,\epsilon) \right)$ $\boxed{\begin{matrix} u_k = \cos\theta_k \\ v_k = \sin\theta_k \end{matrix}}$

$$A_{\uparrow\uparrow}(k,\epsilon) = -2\,\mathrm{Im}\, G_{11}(k, \epsilon^{+i\eta}) \Rightarrow 2\pi\left[u_k^2\,\delta(\varepsilon - E_k) + v_k^2\,\delta(\varepsilon + E_k)\right]$$

$$A_{\downarrow\downarrow}(k,\epsilon) = +2\,\mathrm{Im}\, G_{22,\mathrm{ret}}(k, -\epsilon_{-i\eta}) \Rightarrow 2\pi\left[v_k^2\,\delta(-\varepsilon - E_k) + u_k^2\,\delta(-\varepsilon + E_k)\right]$$

$$\Rightarrow g(\epsilon) = 2\int \frac{d^3k}{(2\pi)^3} \left[u_k^2\,\delta(\varepsilon - E_k) + v_k^2\,\delta(\varepsilon + E_k)\right]$$

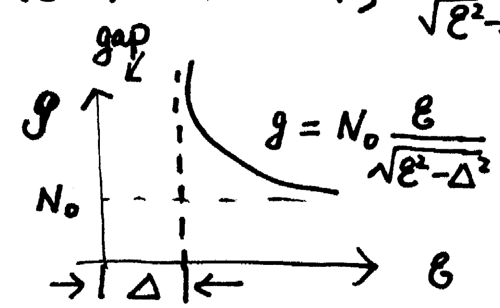
$$= N_0 \int d\xi \left\{ \frac{1}{2}\left[1 + \frac{\xi}{E}\right]\delta(\varepsilon - E) + \frac{1}{2}\left[1 - \frac{\xi}{E}\right]\delta(\varepsilon + E) \right\}$$

where $E = \sqrt{\xi^2 + \Delta^2}$ $\Rightarrow \delta(\varepsilon - \sqrt{\xi^2+\Delta^2}) = \dfrac{\delta(\xi - \sqrt{\varepsilon^2 - \Delta^2}) + \delta(\xi + \sqrt{\varepsilon^2-\Delta^2})}{\frac{|\xi|}{\sqrt{\xi^2+\Delta^2}}}$

$g(\epsilon) = g(-\epsilon)$, so let us consider $\epsilon > 0$

$$g(\varepsilon > 0) = N_0 \int d\xi\, \frac{1}{2}\left(1 + \frac{\xi}{\epsilon}\right)\left[\delta(\xi - \sqrt{\varepsilon^2 - \Delta^2}) + \delta(\xi + \sqrt{\epsilon^2 - \Delta^2})\right] \cdot \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}}$$

$$= N_0 \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}}$$



§: Unconventional Cooper pairing

① spinless fermion p-wave (2D)

$$g(k,k') = g(\vec{k}\cdot\vec{k}') = \frac{g}{2}\left[(k_x+ik_y)(k_x'-ik_y') + (k_x-ik_y)(k_x'+ik_y')\right]$$

$$\Rightarrow \quad \Delta_k = \Delta_0(k_x+ik_y) \leftarrow \text{ or its TR partner}$$

$$\Rightarrow H(k) = \xi_k \tau_3 + \Delta_0 k_x \tau_1 + \Delta_0 k_y \tau_2 \Rightarrow E_k = \sqrt{\xi_k^2 + \Delta_0^2(k_x^2+k_y^2)}$$

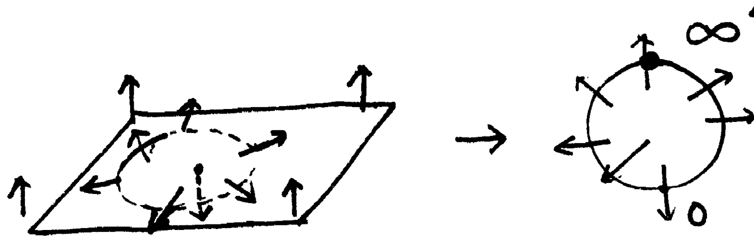
no nodes

topological trivial / non-trivial

① if $\mu > 0$ (within band), $\xi_k < 0$ at $(k_x,k_y)=0$, but $\to \infty$ at $(k_x k_y) \to \infty$

$\Rightarrow$ configuration of $B(k) = (\Delta_0 k_x, \Delta_0 k_y, \xi_k)$

$\Rightarrow$



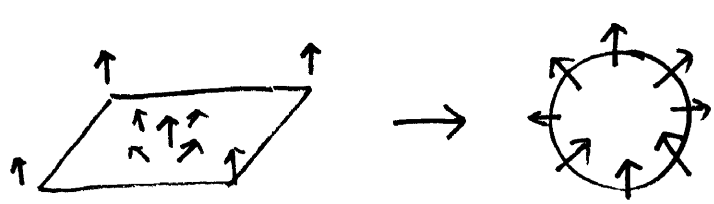
$$\int \frac{dk_x dk_y}{4\pi} \hat{B}\cdot(\partial_x \hat{B} \times \partial_y \hat{B})$$

$= \text{integer}$ $\leftarrow$ Winding number

± 1

② if $\mu < 0$, ξ_k always > 0.

$B(k)$ config



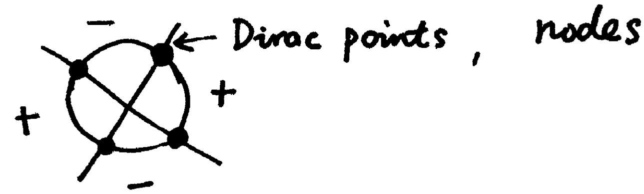
winding number 0.

For topological configuration, $\to$ majarano fermion mode, vortex core, etc ...

②: $d_{x^2-y^2}$ wave $\quad \Delta_k = \Delta_0(\hat{k}_x^2 - \hat{k}_y^2) = \Delta_0 \cos 2\varphi_k$

$$H = \xi_k \tau_3 + \Delta_0(\hat{k}_x^2 - \hat{k}_y^2)\tau_1 . \Rightarrow E_k = \pm\sqrt{v_F^2(k-k_f)^2 + \Delta_0^2 \cos^2 2\varphi_k}$$

Dirac points



$$\text{DOS} \quad g(\epsilon) = 2\int \frac{k\,dk\,d\varphi_k}{(2\pi)^2} \left\{ u_k^2\, \delta(\mathcal{E} - \sqrt{\xi^2 + \Delta^2\cos^2 2\varphi_k}) + v_k^2\, \delta(\mathcal{E} + \sqrt{\xi^2 + \Delta^2\cos^2 2\varphi_k}) \right.$$

for $\mathcal{E} > 0$
$$= N_0 \int d\xi \frac{d\varphi_k}{2\pi} \frac{1}{2}\left(1 + \frac{\xi}{\epsilon}\right) \delta(\epsilon - \sqrt{\xi^2 + \Delta^2\cos^2 2\varphi_k})$$

$$= N_0 \int \frac{d\varphi_k}{2\pi} d\xi \frac{1}{2} \delta(\epsilon - \sqrt{\xi^2 + \Delta^2\cos^2 2\varphi_k}) \qquad \xi_0 = \pm\sqrt{\epsilon^2 - \Delta^2\cos^2 2\varphi_k}$$

$$g(\epsilon) = \frac{N_0}{2\pi}\int d\varphi_k\, d\xi\, \frac{1}{2}\left[\delta(\xi + \xi_0(\varphi_k)) + \delta(\xi - \xi_0(\varphi_k))\right] \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2\cos^2 2\varphi_k}}$$

if $\epsilon > \Delta \Rightarrow g(\epsilon) = N_0 \int \frac{d\varphi_k}{2\pi} \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2\cos^2 2\varphi_k}} = N_0 \int \frac{d\varphi_k}{2\pi} \frac{1}{\sqrt{1 - (\frac{\epsilon}{\Delta})^2\cos^2 2\varphi_k}}$

$\mathcal{E} < \Delta \Rightarrow g(\epsilon) = N_0 \int \frac{d\varphi_k}{2\pi} \frac{\epsilon/\Delta}{\sqrt{(\frac{\mathcal{E}}{\Delta})^2 - \cos^2 2\varphi_k}}$

$|\cos 2\varphi| < \left(\frac{\mathcal{E}}{\Delta}\right)$