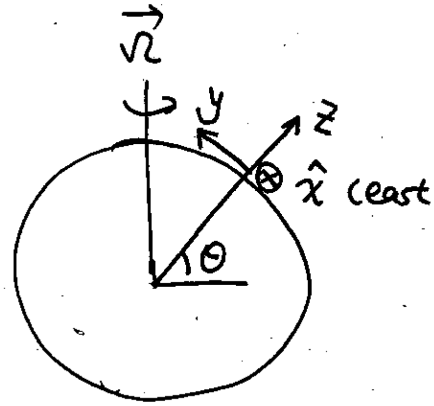
Mid term Solution



Prob 1

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_c$$

Set the frame $\hat{z}$ as the vertically up

$\hat{x}$ as the east

$\hat{y}$ as the north,

θ is the lattitude angle. The earth spinning angular velocity

$$\vec{\Omega} = (0,\ \Omega\cos\theta,\ \Omega\sin\theta)$$

(+2) $$\dot{\vec{r}}\times\vec{\Omega} = (\dot{y}\Omega\sin\theta - \dot{z}\Omega\cos\theta,\ -\dot{x}\Omega\sin\theta,\ -\dot{x}\Omega\cos\theta)$$

$$\Rightarrow \quad \ddot{\vec{r}} = -g\hat{z} + 2\dot{\vec{r}}\times\vec{\Omega} \quad \Rightarrow \quad \begin{cases} \ddot{x} = 2\Omega(\dot{y}\sin\theta - \dot{z}\cos\theta) \\ \ddot{y} = -2\Omega\dot{x}\sin\theta \\ \ddot{z} = -g + 2\Omega\dot{x}\cos\theta \end{cases}$$

1)

• Zero th order solution

(+3) $$\begin{cases} \ddot{x}_0 = \ddot{y}_0 = 0 \\ \ddot{z}_0 = -g \end{cases} \Rightarrow \quad \begin{aligned} & x_0(t) = y_0(t) = 0 \\ & z_0(t) = v_0 t - \tfrac{1}{2} g t^2 \\ & \text{with } v_0^2 = 2gh \end{aligned}$$

and the time for the ball return to the ground is $T = 2\sqrt{2h/g}$.

• 1st order $\ddot{x}_1 = -2\Omega\cos\theta\ \dot{z}_0 = -2\Omega\cos\theta\,[v_0 - gt]$

(+3) with $x_1(0) = 0$, $\dot{x}_1(0) = 0$

$$\Rightarrow \quad x_1 = -2\Omega\cos\theta\left[\frac{v_0}{2}t^2 - \frac{g}{6}t^3\right]$$

$$\Rightarrow \quad x_1 = -\Omega\cos\theta\left(v_0 - \frac{g}{3}t\right)t^2$$

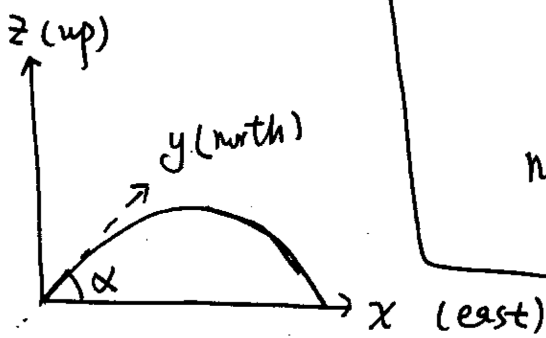
plug in $t = T = \sqrt{\frac{8h}{g}}$, $v_0 = \sqrt{2gh}$

$$\Rightarrow \quad x_1 = -\Omega\cos\theta\,\frac{v_0}{3}\,t^2 \quad (+2)$$

$$= -\Omega\cos\theta\,\frac{1}{3}\sqrt{2gh}\,\frac{8h}{g}$$

$$= -\frac{8\Omega}{3}\sqrt{\frac{2h^3}{g}}\,\cos\theta$$

2)



negative, deviation toward west.

• Zeroth order

$$\begin{cases}\ddot{x}_0 = \ddot{y}_0 = 0 \\ \ddot{z}_0 = -g\end{cases} \quad \text{with} \quad \dot{z}(0) = v_0\sin\alpha, \quad \dot{x}(0) = v_0\cos\alpha$$

(+2)

$$\Rightarrow \begin{cases} x_0(t) = v_0(\cos\alpha)\,t \\ y_0(t) = 0 \\ z(t) = v_0(\sin\alpha)\,t - \frac{1}{2}gt^2 \end{cases}$$

the flying time is

$$T = \frac{2v_0\sin\alpha}{g}$$

• Consider the transverse deviation — correct to 1st order

(+2)

$$\ddot{y}_1 = -2\Omega\,\dot{x}_0\,\sin\theta = -2\Omega v_0\cos\alpha\,\sin\theta$$

plug in $y_1(0) = 0$, $\dot{y}_1(0) = 0$

$$\Rightarrow \quad y_1(t) = -\Omega v_0\cos\alpha\,\sin\theta\,t^2 \quad \Rightarrow \quad y(T) = -\Omega v_0\cos\alpha\,\sin\theta\,\frac{(2v_0\sin\alpha)^2}{g^2}$$

$$= -\frac{4v_0^3\sin^2\alpha\cos\alpha\sin\theta}{g^2}$$

negative, deviation towards south.

(+2)

Prob 2

① $I_{ij} = \int dV \, \rho(\vec{r}) (r^2 \delta_{ij} - r_i r_j)$

after the rotation $r_i' = T_{ij} r_j$

$$I'_{ij} = \int dV \, \rho'(\vec{r}') (r'^2 \delta_{ij} - r_i' r_j')$$ +4

according to $\rho'(\vec{r}') = \rho(r)$, $r'^2 = r^2$ (-2)

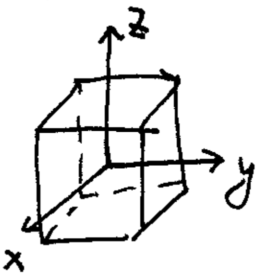
$$\Rightarrow I'_{ij} = \int dV \, \rho(r) (r^2 \delta_{ij} - T_{ii'} r_{i'} T_{jj'} r_{j'})$$

$$= T_{ii'} \Big[\underbrace{\int dV \, \rho(r) (r^2 \delta_{i'j'} - r_{i'} r_{j'})}_{I_{i'j'}} \Big] T^T_{j'j}$$

we have used $T_{ii'} \delta_{i'j'} T^T_{i'j} = \delta_{ij}$ in the above expression

$$\Rightarrow I'_{ij} = T_{ii'} I_{i'j'} T^T_{j'j} \quad \text{or} \quad \boxed{I' = T I T^{-1}}$$

② First we choose xyz axis in the symmetric position



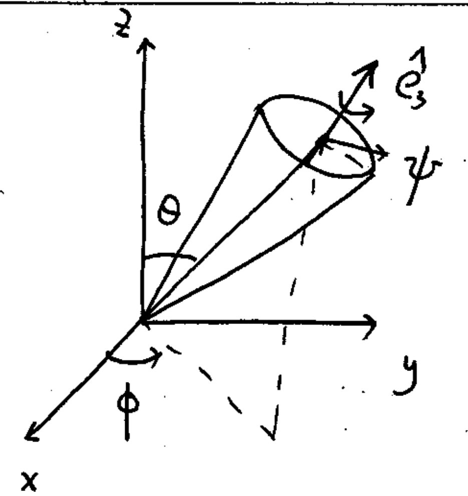
(+2) we have $I_{xy} = I_{yz} = I_{zx} = 0$

$$I_{xx} = I_{yy} = I_{zz} = \int_{-a/2}^{a/2} dx dy dz \, \rho (y^2 + z^2)$$

$$= \rho \cdot a^2 \cdot \left[\frac{y^3}{3} \right] \Bigg|_{-a/2}^{a/2} \times 2 = 2 \rho a^2 \frac{a^3}{12} = \frac{M}{6} a^2 \Rightarrow I = \frac{Ma^2}{6} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

(+2) $\Rightarrow$ under an arbitrary frame $I' = T I T^{-1} = I = \frac{Ma^2}{6} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$.

Prob 3



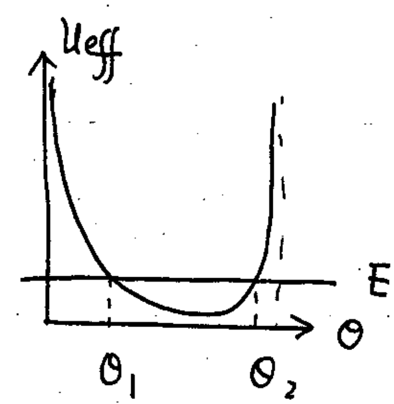
+3

1) at $t=0$, we have $\vec{L} = \lambda_3 \omega \hat{e}_3$

$\Rightarrow L_3 = \lambda_3 \omega$

$L_z = \vec{L} \cdot \hat{z} = \lambda_3 \cos\theta_1$

$$\Rightarrow E = \frac{1}{2}\lambda\dot{\theta}^2 + \frac{(\lambda_3\omega)^2(\cos\theta_1 - \cos\theta)^2}{2\lambda\sin^2\theta} + \frac{L_3^2}{2\lambda_3} + MgR\cos\theta$$



because at $t=0$, $\theta(t=0) = \theta_1$ and $\dot{\theta}(t=0) = 0$

$\Rightarrow \theta_1$ is the turning point with

$$E = U_{eff}(\theta_1) = \frac{L_3^2}{2\lambda_3} + MgR\cos\theta_1$$

+4

2) at another turning point θ_2, we have $\dot{\theta} = 0$

$$\Rightarrow U_{eff}(\theta_2) = U_{eff}(\theta_1)$$

$$\Rightarrow \frac{(\lambda_3\omega)^2[\cos\theta_1 - \cos\theta_2]^2}{2\lambda\sin^2\theta_2} + MgR\cos\theta_2 = MgR\cos\theta_1$$

$$\Rightarrow \frac{(\lambda_3\omega)^2(\cos\theta_1 - \cos\theta_2)}{2MgR\lambda} = \sin^2\theta_2 = 1 - \cos^2\theta_2$$

$$\Rightarrow 1 - \cos^2\theta_2 - p(\cos\theta_1 - \cos\theta_2) = 0$$

+3

define $x = \cos\theta_1 - \cos\theta_2 \Rightarrow \cos\theta_2 = \cos\theta_1 - x$

$$\cos^2\theta_2 = \cos^2\theta_1 + x^2 - 2\cos\theta_1\, x$$

$$\Rightarrow 1 - \cos^2\theta_1 - x^2 + 2\cos\theta_1\, x - p\,x = 0$$

$$\Rightarrow x^2 + (p - 2\cos\theta_1)\,x - \sin^2\theta_1 = 0 \quad \text{if}$$

$$\Rightarrow x = \frac{1}{2}\left[-p + 2\cos\theta_1 \pm \sqrt{(p-2\cos\theta_1)^2 + 4\sin^2\theta_1}\right]$$

we take the value $x > 0$

i.e. $\cos\theta_1 > \cos\theta_2$

(x1)

$$\simeq \frac{1}{2}\left[-(p-2\cos\theta_1) + (p-2\cos\theta_1)\left[1 + \frac{4\sin^2\theta_1}{(p-2\cos\theta_1)^2}\cdot\frac{1}{2}\right]\right]$$

$$\simeq \frac{1}{2}\cdot\frac{4\sin^2\theta_1}{p-2\cos\theta_1}\cdot\frac{1}{2} \simeq \frac{\sin^2\theta_1}{p-2\cos\theta_1} \simeq \frac{\sin^2\theta_1}{p}$$

(x2)

3) $\Rightarrow \cos\theta_1 - \cos\theta_2 \simeq \sin^2\theta_1 \dfrac{2\lambda M g R}{\lambda_3^2\omega^2}$

as time evolves, due to resistance, ω decreases

thus nutation becomes more ane more important.

(x3)

4): $\dot\phi = \dfrac{\lambda_3\omega(\cos\theta_1 - \cos\theta)}{\lambda\sin^2\theta}$. we replace $\overline{\cos\theta_1 - \cos\theta}$ as $\frac{1}{2}(\cos\theta_1 - \cos\theta_2) = \frac{x}{2}$ at $p \gg 1$.

we also replace $\overline{\sin^2\theta} = \sin^2\theta_1$ at $p \gg 1$

Same as we get in the limit $p \gg 1$.

$$\Rightarrow \overline{\dot\phi} = \frac{\lambda_3\omega x}{2\lambda\sin^2\theta_1} = \frac{\lambda_3\omega}{2p\lambda} = \frac{\lambda_3\omega}{(\lambda_3\omega)^2/Mgl} = \frac{Mgl}{\lambda_3\omega}$$