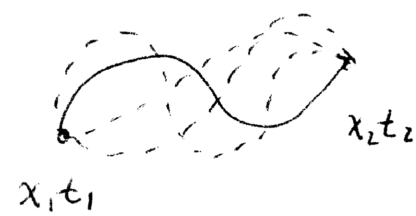
Lect 8 Hamiltonian mechanics (I)

★ motivation / background

Newtonian mechanics : $F = ma$ ← not convenient for quantum mechanics

Lagrangian : → Least action principle



$$\sum_{[X(t)]} e^{i S[X(t)]}$$

↖ path integral of quantum mechanics

$\mathcal{L}(x, \dot{x}(t), t)$ more convenient for Quantum field theory

Hamiltonian $H(q, p)$ ⟶ operator formalism of QM

$\{q, p\} = 1$ $\qquad [x, p] = i\hbar$

★ From Lagrangian to Hamiltonian

$$\mathcal{L} = \mathcal{L}(q_1, \cdots q_n ; \dot{q}_1 \cdots \dot{q}_n , t) = T - U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)$$

define canonical momentum $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$

By Lendgre transformation. define

$$H = \sum_i P_i \dot{q}_i - \mathcal{L}$$ ← we treat (q_i, P_i) as fundamental variables

From $P_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}(q_i\, \dot{q}, t) \underset{\text{solve}}{\Rightarrow} \dot{q}_i = \hat{q}_i(q_i, P_i, t)$

Ex a simple example: free particle:

$$\boxed{\begin{aligned} &\mathcal{L} = T = \frac{1}{2} m \dot{x}^2 \\ &P = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{P}{m} \\ &H = \dot{x}P - \mathcal{L} = \frac{P^2}{m} - \frac{1}{2}\frac{P^2}{m} = \frac{P^2}{2m} \end{aligned}}$$

§ Hamiltonian equation

$$H = P\dot{q} - \mathcal{L} \qquad \text{where } \dot{q} = \dot{q}(q, P)$$
$$\mathcal{L} = \mathcal{L}(q, \dot{q}(q,P), t)$$

$$\frac{\partial H}{\partial P} = \dot{q} + P\frac{\partial \dot{q}}{\partial P} - \frac{\partial \mathcal{L}}{\partial P} = \dot{q} + P\frac{\partial \dot{q}}{\partial P} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial P} = \dot{q}$$

$$\frac{\partial H}{\partial q} = P\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial \mathcal{L}}{\partial q} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}\right] = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] = -\dot{P}$$

$$\Rightarrow \boxed{\frac{\partial H}{\partial P} = \dot{q}, \quad \frac{\partial H}{\partial q} = -\dot{P}}$$

Generalization to multiple - dimension / degrees of freedom

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t), \quad H = \sum P_i \dot{q}_i - \mathcal{L}$$

$$\rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial H}{\partial q_i}.$$

★ <u>time - derivatie : Poisson braket</u>

Any physical quantity $O(q_i, P_i, t)$.

$$\frac{d}{dt} O(q_i(t), P_i(t), t) = \sum_i \frac{\partial O}{\partial q_i} \dot{q}_i + \frac{\partial O}{\partial P_i} \dot{P}_i + \frac{\partial O}{\partial t}$$

$$= \sum_i \left\{ \frac{\partial Q}{\partial q_i} \frac{\partial H}{\partial P_i} - \frac{\partial Q}{\partial P_i} \frac{\partial H}{\partial q_i} \right\} + \frac{\partial Q}{\partial t}$$

$$= \{O, H\}_p + \frac{\partial Q}{\partial t}$$

$$\{O_1, O_2\}_p = \sum_i \left\{ \frac{\partial O_1}{\partial q_i} \frac{\partial O_2}{\partial P_i} - \frac{\partial O_1}{\partial P_i} \frac{\partial O_2}{\partial q_i} \right]$$

easy to check $\quad [O_1, O_2]_p = -[O_2, O_1]_p$

$$[H, H]_p = 0$$

$$[q_i, P_i]_p = 1$$

$$\Rightarrow \quad \frac{d}{dt} H = \frac{\partial H}{\partial t}.$$

Conserved quantity $\quad O(q_i, P_i) \iff [O, H]_p = 0$

O doesn't depend on "t" explicitly.

Example: Hamiltonian Equation in a center force field

central force $\longleftrightarrow$ motional is co-planen,

use ploar coordinates:

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) \qquad \mathcal{L} = T - U(r)$$

$$P_r = \frac{\partial \mathcal{L}}{\partial \dot{r}} = m \dot{r} \quad \Rightarrow \quad \dot{r} = \frac{P_r}{m}$$

$$P_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi} \qquad \dot{\phi} = \frac{P_\phi}{m r^2}$$

$$H = \cancel{\Sigma}\, P_r \dot{r} + P_\phi \dot{\phi} - T + U(r)$$

$$= \frac{1}{2} \frac{P_r^2}{m} + \frac{1}{2} \frac{P_\phi^2}{m r^2} + U(r)$$

centrifugal term

$$\Rightarrow \quad \dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r}{m}, \qquad \boxed{\dot{P}_r = -\frac{\partial H}{\partial r} = -\frac{P_\phi^2}{m r^3} - \frac{dU}{dr}}$$

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{m r}, \qquad \dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

$$P_\phi = l_z \qquad \dot{l}_z = 0$$

angular momentum conservation!

ignorable variables: if H does not dependent on q_i

$$\Rightarrow \boxed{\dot{P}_i = -\frac{\partial H}{\partial q_i} = 0 \Rightarrow P_i = \text{const}}$$