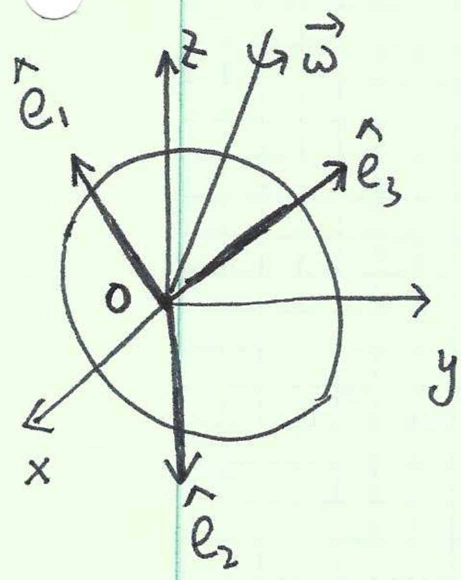
Lecture 6 Euler equation

space frame : inertial frame with fixed xyz-axes

body frame : principle axes — body axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$

rotating frame / non-inertial frame

If the body has a fixed point, we set the origin "O" at that point, then we do not need to worry the force exerted by "O". If the body has no fixed point, we choose the origin of the frames "O" at center of mass.



If the rigid body rotates at angular velocity $\vec{\omega}$.

Then we write down

$$\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$$

in the body frame.

$$\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma} \leftarrow \text{torque} = \Gamma_1 \hat{e}_1 + \Gamma_2 \hat{e}_2 + \Gamma_3 \hat{e}_3$$

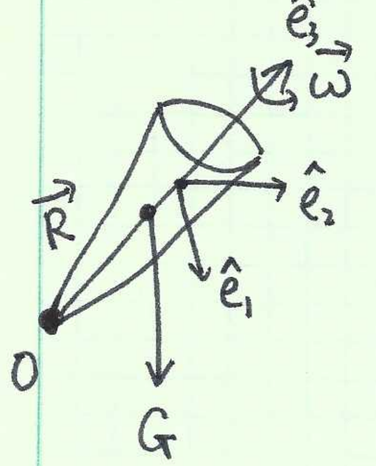
$$= \lambda_1 \dot{\omega}_1 \hat{e}_1 + \lambda_2 \dot{\omega}_2 \hat{e}_2 + \lambda_3 \dot{\omega}_3 \hat{e}_3 + \vec{\omega} \times (\lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3) = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

$$\vec{\omega} \times \vec{L} = (\omega_1 l_2 - \omega_2 l_1)\hat{e}_3 + (\omega_2 l_3 - \omega_3 l_2)\hat{e}_1 + (\omega_3 l_1 - \omega_1 l_3)\hat{e}_2$$

$$= \omega_1\omega_2(\lambda_2-\lambda_1)\hat{e}_3 + \omega_2\omega_3(\lambda_3-\lambda_2)\hat{e}_1 + \omega_3\omega_1(\ell_1-\ell_3)\hat{e}_2$$

$$\Rightarrow \boxed{\begin{aligned} \lambda_1\dot{\omega}_1 - (\lambda_2-\lambda_3)\omega_2\omega_3 &= \Gamma_1 \\ \lambda_2\dot{\omega}_2 - (\lambda_3-\lambda_1)\omega_3\omega_1 &= \Gamma_2 \\ \lambda_3\dot{\omega}_3 - (\lambda_1-\lambda_2)\omega_1\omega_2 &= \Gamma_3 \end{aligned}}$$

Application: $\Gamma_{1,2,3}$ are components in the rotating frame, and thus is difficult to trace. Euler's equation is mostly useful for $\vec{\Gamma}=0$. or, for the case one of it's component is zero



symmetric top spinning in the gravity field

$$\vec{\Gamma} = \vec{R} \times \vec{G}$$

$$= R\,\hat{e}_3 \times \vec{G}$$

$\Rightarrow \vec{\Gamma}\cdot\hat{e}_3 = 0$, also because $\lambda_1 = \lambda_2$.

$$\Rightarrow \boxed{\lambda_3\dot{\omega}_3 = 0}$$

§ Euler equation with zero torque

$\vec{L}$ is conserved in the lab frame, but not in the body frame.

$$\boxed{\begin{aligned} \lambda_1 \dot{\omega}_1 &= (\lambda_2 - \lambda_3)\,\omega_2 \omega_3 \\ \lambda_2 \dot{\omega}_2 &= (\lambda_3 - \lambda_1)\,\omega_3 \omega_1 \\ \lambda_3 \dot{\omega}_3 &= (\lambda_1 - \lambda_2)\,\omega_1 \omega_2 \end{aligned}}$$

★ if we start with $\vec{\omega}$ // a principle axis, say $\hat{e}_1$

$$\Rightarrow \omega_2 = \omega_3 = 0 \text{ at } t = 0 \Rightarrow \begin{cases} \lambda_1 \dot{\omega}_1 = 0 \\ \lambda_2 \dot{\omega}_2 = 0 \\ \lambda_3 \dot{\omega}_3 = 0 \end{cases}$$

$$\Rightarrow \dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$$

$\vec{\omega}$ is a constant.

★ if $\vec{\omega}$ is not along any principle axis, and $\lambda_1 \neq \lambda_2 \neq \lambda_3$ then at most only one of $\omega_{1,2,3}$ can be nonzero, we have $\dot{\vec{\omega}} \neq 0$. Thus although $\vec{L}$ is conserved, but $\vec{\omega}$ is not!

★ Next we study the stability ~~pro~~ of rotation around principle axis.

Suppose at $t=0$, $\vec{\omega} = \omega_3 \hat{e}_3$ with $\omega_1 = \omega_2 = 0$. Then we give a small perturbation ω_1 and ω_2, will $\omega_{1,2}$ grow larger and larger?

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\, \omega_1 \omega_2 \sim \text{second order small quantity}$$
$$\sim 0$$

$\Rightarrow \omega_3 \sim$ const.

linearized Euler equation $\quad \dfrac{d}{dt}\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_3 \\ \frac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3 & 0 \end{bmatrix}}_{A} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$

try solution $\quad \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\mu t}$

$$\Rightarrow \mu \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\mu t} = A \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\mu t} \Rightarrow [A - \mu I] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$\Rightarrow \mu$ is A's eigenvalue and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is eigenvector.

$$A - \mu I = \begin{bmatrix} -\mu & \frac{\lambda_2 - \lambda_3}{\lambda_1}\omega_3 \\ \frac{\lambda_3 - \lambda_1}{\lambda_2}\omega_3 & -\mu \end{bmatrix}, \quad \det(A - \mu I) = 0 \Rightarrow$$

$$\mu^2 = -\frac{\omega_3^2}{\lambda_1 \lambda_2}(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3).$$

* if λ_3 is between $\lambda_{1,2}$, then $\mu^2 > 0 \Rightarrow$

there's a pair of positive & negative eigenvalues

$$\mu_{1,2} = \pm \frac{\omega_3}{\sqrt{\lambda_1 \lambda_2}} \sqrt{(\lambda_1 - \lambda_3)(\lambda_3 - \lambda_2)}$$

and $\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix}(t) \propto \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\mu_1 t} + \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} e^{\mu_2 t}$. The positive eigenvalue diverges. unstable!

★ if λ_3 is the largest or smallest

$$\Rightarrow \mu = \pm \frac{i\,\omega_3}{\sqrt{\lambda_1\lambda_2}} \sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \Rightarrow \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ oscillates with time.}$$

We solve eigenvector

$$\begin{bmatrix} -\mu & \frac{\lambda_2-\lambda_3}{\lambda_1}\omega_3 \\ \frac{\lambda_3-\lambda_1}{\lambda_2}\omega_3 & -\mu \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} \frac{\mp i}{\sqrt{\lambda_1\lambda_2}}\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} & \frac{\lambda_2-\lambda_3}{\lambda_1} \\ \frac{\lambda_3-\lambda_1}{\lambda_2}, & \frac{\mp i}{\sqrt{\lambda_1\lambda_2}}\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0$$

$$\Rightarrow \pm i \frac{C_1}{\sqrt{\lambda_1\lambda_2}} \sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)} = \frac{\lambda_2-\lambda_3}{\lambda_1} C_2$$

$$\Rightarrow C_2 = \pm i \sqrt{\frac{\lambda_1}{\lambda_2}\frac{\lambda_1-\lambda_3}{\lambda_2-\lambda_3}}\ \mathrm{sgn}(\lambda_2-\lambda_3)\ C_1$$

$$\Rightarrow \begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \text{Const.} \begin{pmatrix} 1 \\ \pm i \sqrt{\frac{\lambda_1}{\lambda_2}\frac{(\lambda_1-\lambda_3)}{(\lambda_2-\lambda_3)}}\ \mathrm{sgn}(\lambda_2-\lambda_3) \end{pmatrix} e^{\pm i|\mu|t}$$

use the real number, and set $\omega_1(t) \propto \cos|\mu|t$, we have

$$\begin{pmatrix} \omega_1(t) \\ \omega_2(t) \end{pmatrix} = \text{Const} \begin{pmatrix} (e^{i|\mu|t} + e^{-i|\mu|t}) \\ \sqrt{\frac{\lambda_1(\lambda_1-\lambda_3)}{\lambda_2(\lambda_2-\lambda_3)}}\ \mathrm{sgn}(\lambda_2-\lambda_3)\ i\,(e^{i|\mu|t} - e^{-i|\mu|t}) \end{pmatrix}$$

$$= \text{Const.} \begin{pmatrix} \cos|\mu|t \\ -\sqrt{\frac{\lambda_1}{\lambda_2}\frac{(\lambda_1-\lambda_3)}{(\lambda_2-\lambda_3)}}\ \mathrm{sgn}(\lambda_2-\lambda_3)\ \sin|\mu|t \end{pmatrix}$$

* For symmetric top $\lambda = \lambda_1 = \lambda_2 \Rightarrow$ $\omega_3 = \text{const}$ exactly

and $\mu = \pm i \dfrac{\omega_3}{\lambda} |\lambda - \lambda_3|$

the above solution: $\vec{\omega} = \begin{pmatrix} \omega_0 \cos\left(\dfrac{\omega_3(\lambda-\lambda_3)}{\lambda} t\right) \\ -\omega_0 \sin\left(\dfrac{\omega_3(\lambda-\lambda_3)}{\lambda} t\right) \\ \omega_3 \end{pmatrix}$

$\text{sgn}(\lambda-\lambda_3) \sin\left(\dfrac{\omega_3 |\lambda-\lambda_3| t}{\lambda}\right)$

$= \sin \dfrac{\omega_3(\lambda-\lambda_3)t}{\lambda}$

* Consider an "egg-shapped" config $\lambda_3 < \lambda_1 = \lambda_2 = \lambda$

$\vec{\omega} = \omega_0 \cos\Omega_b t \, \hat{e}_1 - \omega_0 \sin\Omega_b t \, \hat{e}_2 + \omega_3 \hat{e}_3$

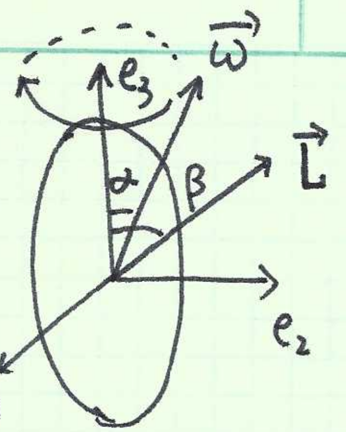
where $\Omega_b = \dfrac{\omega_3}{\lambda}(\lambda - \lambda_3)$

In the body frame $\vec{\omega}$ precess around $-\hat{e}_3$

$\vec{L} = \lambda_1 \omega_1 \hat{e}_1 + \lambda_2 \omega_2 \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$

$= \lambda \omega_0 \cos\Omega_b t \, \hat{e}_1 - \lambda \omega_0 \sin\Omega_b t \, \hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$

$\Rightarrow \vec{L}$ precesses around $-\hat{e}_3$ with the same angular velocity Ω_b

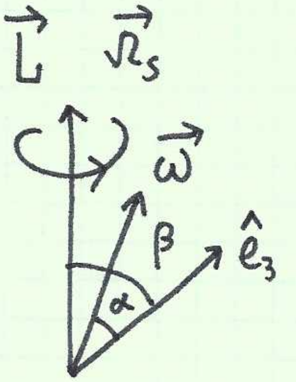


$\hat{e}_3, \vec{\omega}, \vec{L}$ coplaner

$$\tan\alpha = \frac{\omega_0}{\omega_3}$$

$$\tan\beta = \frac{\omega_0}{\omega_3}\frac{\lambda}{\lambda_3} \Rightarrow \beta > \alpha .$$

how about the motion in the lab frame? $\vec{L}$ is fixed



$\hat{e}_3, \vec{\omega}$ must precess around $\vec{L}$

we assume it's angular velocity is $\vec{\Omega}_s$:

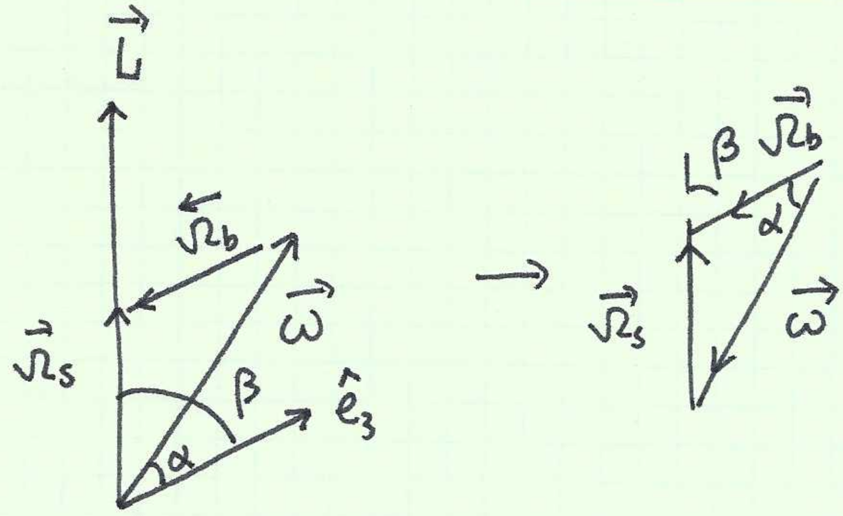
$\vec{\omega}$ precesses around $\hat{e}_3$ with angular velocity $\vec{\Omega}_b = -\Omega_b \hat{e}_3$ is the body [illegible] and the frame

body frame is rotating [illegible] at angular velocity $\vec{\omega}$ in the Lab frame

$$\Rightarrow \vec{\Omega}_s = \vec{\Omega}_b + \vec{\omega} \longleftarrow$$ body frame rotation in the Lab frame

$\uparrow$ $\hat{e}_3$ precess in the Lab frame

$\hat{e}_3$ precession in the body frame



$$\Rightarrow \boxed{\frac{\Omega_s}{\sin\alpha} = \frac{\omega}{\sin\beta}}$$

sine law