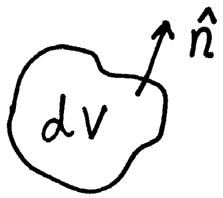
Lect 12 Stress - Strain - Elasticity of Solids (I)

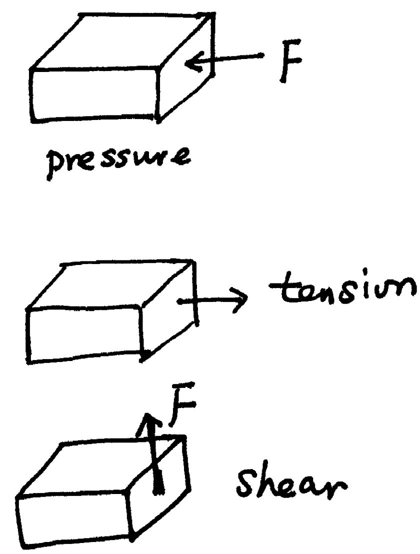
§1 volume forces :



say gravity $\vec{F} = \rho \vec{g}\, dV$

electro-static $\vec{F} = \rho \vec{E}\, dV$

② surface force



The difference between solides & liquids :

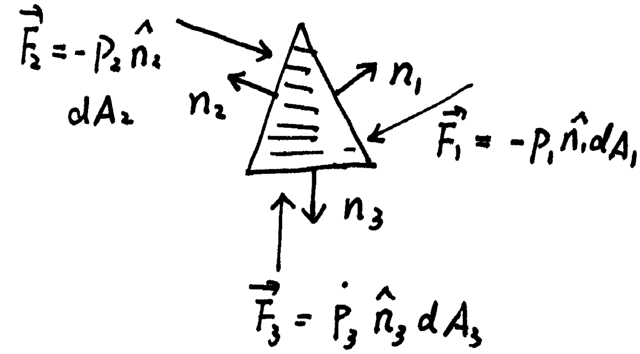
$\boxed{\text{liquids have no shear modulus}}$.

⇒ Isotropic pressure of liquids

Let us choose two surfaces S_1, S_2 normal to arbitrary unit vectors $\hat{n}_1\ \hat{n}_2$.

Find the 3rd surface S_3 with $\hat{n}_3$ ⇒ from an isoseles prism

(There're two ends of the prism, but they are normal to the plane). Consider to in-plane force components.



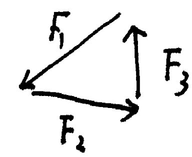
$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_{vol} = m\vec{a}$$

$$\underbrace{\vec{F}_1 + \vec{F}_2 + \vec{F}_3}_{\text{surface forces}} = \underbrace{m\vec{a}}_{\text{volume forces}} - \vec{F}_{vol}$$

Let's do scale transformation; shrink the size by a factor λ

$\Rightarrow \lambda^2 (\vec{F}_1 + \vec{F}_2 + \vec{F}_3) = \lambda^3 (m\vec{a} - \vec{F}_{vol})$, as $\lambda \to 0$

$\Rightarrow \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$. Because $F_{1,2}$ form the same angle with F_3,



$\Rightarrow F_1 = F_2 \Rightarrow$ iso-pressure.
tropic

direct consequences of no-shear modulus.

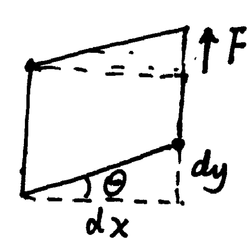
§2. Elastic moduli

Stress: any surface force F proportional the area A

say pressure $p = \frac{F}{A}$, $\frac{\text{tension}}{A}$, $\frac{\text{Shearing force}}{A}$.

Strain: the change the volume, length, ... (fractional deformation)

$\frac{dV}{V}$ (static fluid), $\frac{dl}{l}$ (a wire in tension) $\frac{dy}{dx}$ (for a shear)

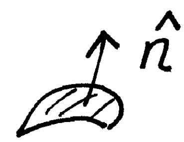


Stress = (Young's modulus) × Stain $\frac{dF}{A} = YM \frac{dl}{l}$

Stress = (bulk modulus) × Stain $\Rightarrow$ $dp = -BM \frac{dV}{V}$

stress = (shear modulus) × stain $\Rightarrow$ $\frac{F}{A} = SM \frac{dy}{dx}$ etc.

§ The stress-tensor



define an orientational area $d\vec{A} = \hat{n}\, dA$

* The surface force acting on the area $d\vec{A}$ is dentoed as $\vec{F}(d\vec{A})$, we will prove $\vec{F}(d\vec{A})$ is a linear function of $d\vec{A}$, i.e. $\vec{F}(\lambda_1 d\vec{A}_1 + \lambda_2 d\vec{A}_2) = \lambda_1 \vec{F}(d\vec{A}_1) + \lambda_2 \vec{F}(d\vec{A}_2)$.

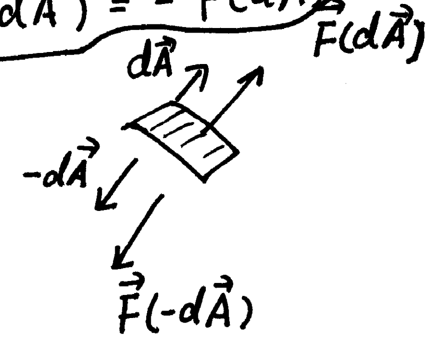
First, as long as for small dA, we have

$$\boxed{\vec{F}(\lambda d\vec{A}) = \lambda F(d\vec{A}), \quad \text{and} \quad \vec{F}(-d\vec{A}) = -\vec{F}(d\vec{A})}$$

$\Uparrow$

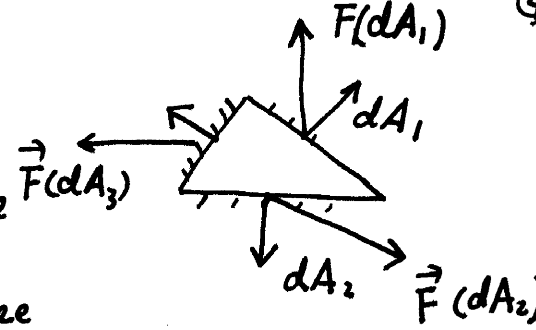
$\vec{F}(d\vec{A}) + \vec{F}(-d\vec{A}) = 0$

Newton's third law.



Second: consider two area elements dA_1 & dA_2,

find a third area element $d\vec{A}_3 = -(d\vec{A}_1 + d\vec{A}_2)$



$$\Rightarrow \quad \vec{F}(dA_1) + \vec{F}(dA_2) + \vec{F}(dA_3) = \underbrace{ma - \vec{F}_{volume}}_{\text{Volume force}}$$

as surface size $\to 0$, right hand side $\to 0$

$$\Rightarrow \quad \vec{F}(dA_1) + \vec{F}(dA_2) + \vec{F}(d\vec{A}_3) = 0$$

i.e. $\vec{F}(d\vec{A}_1 + d\vec{A}_2) = \vec{F}(-d\vec{A}_3) = -\vec{F}(d\vec{A}_3) = \vec{F}(d\vec{A}_1) + \vec{F}(d\vec{A}_2)$

combine with $F(\lambda dA) = \lambda F(dA)$

$$\Rightarrow \quad \boxed{\vec{F}(\lambda_1 d\vec{A}_1 + \lambda_2 d\vec{A}_2) = \lambda_1 \vec{F}(d\vec{A}_1) + \lambda_2 \vec{F}(d\vec{A}_2)}$$

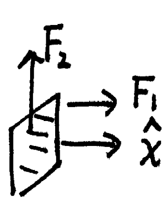
$\vec{F}$ direction isnot necessary along $d\vec{A}$ if not in liquid.

define

$$\boxed{F_i(d\vec{A}) = \sum_{j=1}^{3} \sigma_{ij}\, dA_j \qquad (i, j = \hat{x}, \hat{y}, \hat{z})}$$

↑ 3 × 3 matrix : tensor

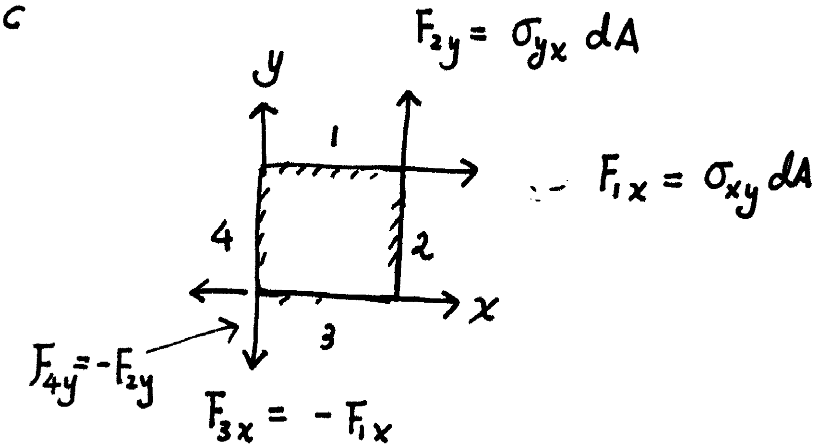
say choose an area element along $\hat{x}$



$$F_1 = \sigma_{11}\, dA$$

$$F_2 = \sigma_{21}\, dA$$

§ Stress-Tensor is symmetric



the torque $F_{2y} \cdot l - F_{1x} \cdot l$

$$= (\sigma_{yx} - \sigma_{xy})\, l\, dA$$

$$= \frac{dL_3}{dt}$$

rescale by a factor λ : LHS gains $\lambda \cdot \lambda^2 = \lambda^3$

RHS gains $\lambda, \lambda^3 = \lambda^4$

$$\Rightarrow \quad \boxed{\sigma_{yx} = \sigma_{xy}}$$

σ_{ij} only has 6- independent elements.

Example: a static fluid :

$\vec{F}$ is along $d\vec{A}$ $\Rightarrow$ $\vec{F}(d\vec{A}) = -p\, d\vec{A}$

$$\Rightarrow \Sigma = -p\, I$$

Generally force does not need to be paralled to surface.

$$\vec{F} = \vec{\vec{\Sigma}}\, \vec{A}$$