Lect 11 Continuum Mechanics - Wave (I)

point masses $\longrightarrow$ rigid body $\longrightarrow$ continuum mechanics

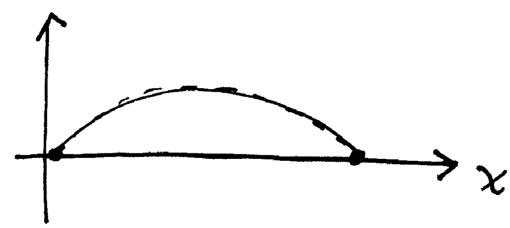
$\underbrace{\hspace{6cm}}_{\text{discrete, number of coordinates}}$ elasticity of solids

fluid mechanics.

ordinary differential equation $\longrightarrow$ partial differential Eq.

§1: Wave equation in one dimenson

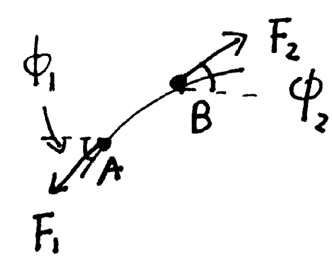
$y = u(x)$



a ~~taut~~ string $y = u(x, t)$

displacement as a function of x, t.

Let us pick up a small segment AB



The magnitude of $F_{1,2}$ are roughly the same : tension T.

$$F_x^{net} = T(\cos\phi_2 - \cos\phi_1) \approx \frac{T}{2}(\phi_2^2 - \phi_1^2) \simeq 0 \text{ (second order)}$$

$$F_y^{net} = T(\sin\phi_2 - \sin\phi_1) \approx T\cos\phi \, d\phi \qquad \begin{pmatrix} \phi_2 = \phi + d\phi \\ \phi_1 = \phi \end{pmatrix}$$

$$\cancel{tg} \tan\phi \approx \phi \approx \frac{\partial u}{\partial x}$$

$$\cos\phi \simeq 1$$

$$F_y^{net} \approx T\, d\phi \approx T \frac{\partial \phi}{\partial x}\, dx = T \frac{\partial^2 u}{\partial x^2}\, dx = \Delta m\, a$$

$$= \cancel{\rho}.\ \mu\, dx \frac{\partial^2 u}{\partial t^2}$$ μ: mass density

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial^2 u}{\partial x^2}}, \text{ where } c = \sqrt{\frac{T}{\mu}}$$

wave equation. C: sound wave velocity. The more tight of the string, the higher of C.

§ Solution of wave equation

introduce $\xi = x - ct$, $\eta = x + ct$ $\Rightarrow$ $x = \frac{\xi + \eta}{2}$, $ct = \frac{-\xi + \eta}{2}$

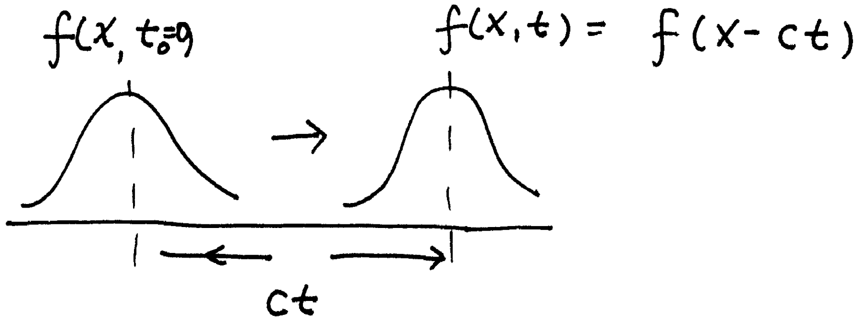
$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial (ct)}\frac{\partial (ct)}{\partial \xi} = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{\partial}{\partial (ct)}\right]$$

$$\frac{\partial}{\partial \eta} = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{\partial}{\partial (ct)}\right]$$

$$\Rightarrow \frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{c^2 \partial t^2}\right] \Rightarrow \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -4c^2 \frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta} = 0$$

$\Rightarrow$ $u = f(\xi) + g(\eta)$, where $f(\xi)$, $g(\eta)$ are arbitary function of ξ, η.

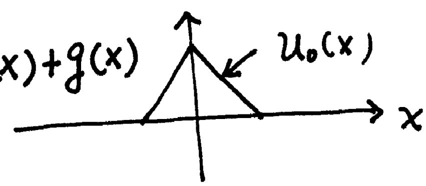
$$= \underset{\substack{\uparrow \\ \text{right mover}}}{f(x - ct)} + \underset{\substack{\uparrow \\ \text{left mover}}}{g(x + ct)}$$



Example: evolution of a triagular wave.

second order differential Eq : $\begin{cases} \text{initial } u(x,t_0) & a) \\ \text{initial velocity } \frac{\partial}{\partial t} u(x,t_0) & b) \end{cases}$

a) $u(x, t_0=0) = u_0(x) = f(x) + g(x)$



b) $\left.\frac{\partial u}{\partial t}\right|_{t=t_0} = 0$

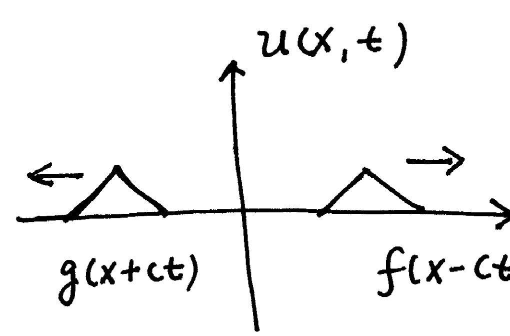
$$\left.\frac{\partial u}{\partial t}\right|_{t=t_0} = \left.\left[\frac{\partial f(x-ct)}{\partial t} + \frac{\partial g(x+ct)}{\partial t}\right]\right|_{t=t_0} = c\left[-\frac{\partial f(x-ct)}{\partial ct} + \frac{\partial g(x+ct)}{\partial ct}\right]$$

$$= c\left.\left[-\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right]\right|_{t=t_0} = 0$$

$\Rightarrow \quad \Rightarrow \quad f(x) - g(x) = \text{const} = C$

$$\Rightarrow \quad f(x) = \frac{u_0(x) + C}{2}$$

$$g(x) = \frac{(u_0(x) - C)}{2}$$



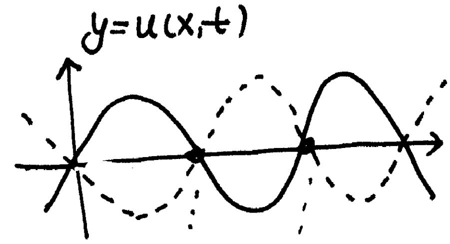
$$\Rightarrow u(x,t) = \left[f(x-ct) + g(x+ct)\right] = \frac{1}{2} u_0(x-ct) + \frac{1}{2} u_0(x+ct)$$

Example: Standing wave

if $u = f(x-ct) + g(x-ct) = A\left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$

$= 2A \sin kx \cos \omega t$

$\Rightarrow$ the nodes do not change.



nodes $x = \frac{n\pi}{k}$

§ Boundary condition: waves on a finite string

- First type boundary condition (Dirichlet)

$u(0,t) = u(L,t) = 0.$

try $u(x,t) = X(x)\cos(\omega t - \delta)$

seprate variables

plug in $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$\Rightarrow \quad -\omega^2 X(x)\cos(\omega t - \delta) = c^2 \frac{d^2 X(x)}{dx^2} \cos(\omega t - \delta)$$

$$\Rightarrow \quad \frac{d^2 X(x)}{dx^2} = -k^2 X(x) \quad \text{where } k = \frac{\omega}{c}$$

$X(x) = B\cos kx + A\sin kx$, $\leftarrow$ Dirichlet BC $\Rightarrow B = 0$

$k_n = \frac{n\pi}{L}$, $n = 1, 2, 3 \ldots$

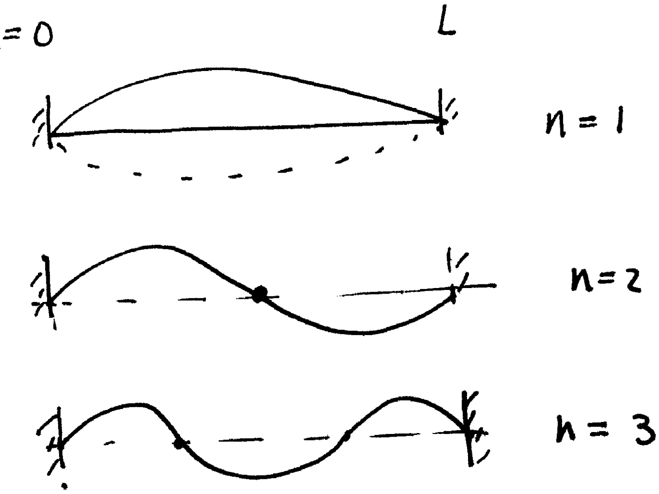
$$\Rightarrow \quad u(x,t) = \sum_n A_n \sin k_n x \cos(\omega_n t - \delta_n)$$

$\omega_n = \frac{n\pi}{L} c$

normal modes (Quantization)

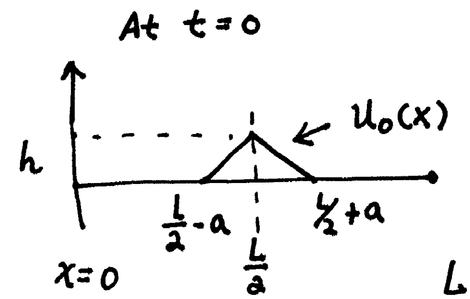
fundamental

overtones



the same for Quantum mechanics particle wave function!

Example: the tringular wave in a finite string.



$$u(x,t) = \sum_n \sin k_n x \,(B_n \cos\omega_n t + C_n \sin\omega_n t)$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_n \sin k_n x \,\omega_n (-B_n \sin\omega_n t + C_n \cos\omega_n t).$$

At $t=0$, $\Rightarrow$ $\frac{\partial u}{\partial t} = 0$ $\Rightarrow$ $\sum_n \omega_n \sin k_n x \cdot C_n = 0$

for all the x $\Rightarrow$ $C_n = 0$.

$$\Rightarrow \quad u(x,t) = \sum_n B_n \sin k_n x \cos\omega_n t.$$

At $t=0$ $\Rightarrow$ $u(x,0) = \sum_n B_n \sin k_n x = u_0(x)$

$$B_n = \frac{2}{L}\int_0^L u_0(x)\sin\frac{\pi n x}{L}\,dx = \frac{2}{L}\int_{-L/2}^{\frac{L}{2}} u_0(x')\sin\left(\frac{n\pi}{L}x' - \frac{n\pi}{2}\right)dx'$$

$$x' = x - \frac{L}{2}$$

$u_0(x')$ is even

$$\Rightarrow \quad B_{2m} = 0$$

$$B_{2m+1} = \frac{4}{L}\int_0^a u_0(x')\,(-)^{m+1}\cos\left(\frac{(2m+1)\pi}{L}x'\right)dx'$$

The fundemental frequency $\omega_1 = \frac{\pi}{L}c$

periodicity $\tau = \frac{2\pi}{\omega_1}$.

static

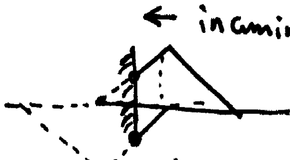
time-evolution: $t=0$

$t = \tau/8$

$t = \tau/4$

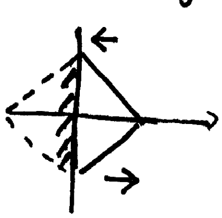
when reach boundary

← incoming



→ reflected

interference

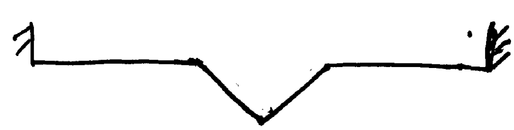


add to zero

$t = \tau/4$

$t = \frac{3\tau}{8}$

$t = \tau/2$



§ 3D wave equation

1D: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow$ 3D $\frac{\partial^2 p}{\partial t} = c^2 \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right)$

p- pressure, sound wave

$c = \sqrt{\frac{BM}{\rho_0}}$ ← bulk modulus (BM), ← density (ρ_0)

plane wave solution: (free - space)

① Certainly $p = f(x-ct) + g(x+ct)$ remains a possible solution, which means its propagation along $\pm \hat{x}$ direction.

Generally speaking, we can choose an arbitary propagation direction $\hat{n}$.

$$p = f(\hat{n} \cdot \vec{r} - ct)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2 = \vec{\nabla} \cdot \vec{\nabla}$$

$$\nabla \cdot p = \frac{df}{d(\hat{n} \cdot \vec{r})} \hat{n} = -\frac{1}{c} \frac{\partial f}{\partial t} \hat{n}, \Rightarrow \nabla(\nabla \cdot p) = -\frac{\hat{n}}{c} \nabla\left(\frac{\partial f}{\partial t}\right)$$

$$\nabla^2 p = -\frac{\hat{n}}{c} \cdot \left(-\frac{\hat{n}}{c}\right) \frac{\partial}{\partial t} \frac{\partial f}{\partial t} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} f$$

In particular, the plane-wave $p = \cos[k(\hat{n} \cdot \vec{r} - \omega t)]$

Spherical coordinate

$$\frac{\nabla^2 f}{f} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

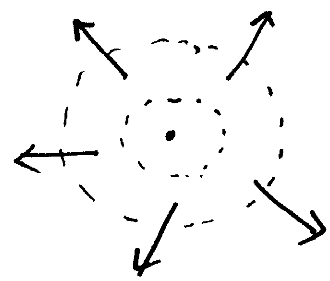
we can choose

$$r\,p(r,t) = f(r-ct) + g(r+ct)$$

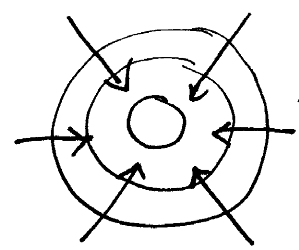
$$\Rightarrow \quad p(r,t) = \underbrace{\frac{1}{r} f(r-ct)}_{\text{out-going spherical wave}} + \underbrace{\frac{1}{r} g(r+ct)}_{\text{in-coming spherical wave}}$$

out-going spherical wave

in-coming spherical wave



$\dfrac{f(r-ct)}{r}$



$\dfrac{g(r+ct)}{r}$

power $\sim \int r^2 dr \left(\frac{1}{r}\right)^2$

no-power divergence.

quantum mechanical

Schrödinger Eq

$$\frac{\partial}{\partial t}\psi(\vec{r},t) = \overbrace{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]}^{\text{Kinetics energy}} \psi(\vec{x},t)$$

$$+ \underbrace{V(\vec{r},t)}_{\text{potential}}\, \psi(\vec{r},t)$$

\+ Boundary ~~eq~~ condition:

Free space $V=0 \Rightarrow$ free-wave equation

$\boxed{\text{Wave mechanics}}$