

Take home final exam- Phys110B, due on March 18 2016, 3:00pm

Please write down a sentence to express that you have done the final exam independently without discussion with others.

You may or may not need to use the following vector algebra formulas

In terms of each component

$$\begin{aligned}(\vec{\nabla} \times (\vec{\nabla} \times \vec{C}))_i &= \partial_i(\partial_j C_j) - \partial_j^2 C_i, \\(\vec{A} \times (\vec{\nabla} \times \vec{B}))_i &= -A_j \partial_j B_i + A_j \partial_i B_j,\end{aligned}\tag{1}$$

where the repeated indices means summation. Or in a compact notation,

$$\begin{aligned}\vec{\nabla} \times (\vec{\nabla} \times \vec{C}) &= \vec{\nabla}(\vec{\nabla} \cdot \vec{C}) - \nabla^2 \vec{C} \\ \vec{A} \times (\vec{\nabla} \times \vec{B}) &= -(\vec{A} \cdot \vec{\nabla})\vec{B} + \nabla_B(\vec{A} \cdot \vec{B}),\end{aligned}\tag{2}$$

where ∇_B only applies on \vec{B} .

Problem 1. Canonical equations (10 points)

Consider a rotating frame whose angular velocity with respect to the inertial frame is $\vec{\Omega}$. We have learned that the Lagrangian of a mass particle m in such a frame is

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2}m\dot{\vec{r}}^2 + m\dot{\vec{r}} \cdot (\vec{\Omega} \times \vec{r}) + \frac{m}{2}(\vec{\Omega} \times \vec{r})^2. \quad (3)$$

a) Start from this Lagrangian, and prove that the corresponding Hamiltonian can be written in such a form as

$$H = \frac{(\vec{P} - m\vec{A}(\vec{r}))^2}{2m} + V(r), \quad (4)$$

Determine the form of $\vec{A}(\vec{r})$ and $V(r)$.

b) Using the Hamiltonian equations to prove the equation of motion as

$$m\ddot{\vec{r}} = m\dot{\vec{r}} \times (\nabla \times \vec{A}(\vec{r})) - \nabla V(r). \quad (5)$$

Hint: you may want to use the formula

$$\frac{d\vec{A}(\vec{r})}{dt} = \frac{\partial \vec{A}(\vec{r})}{\partial t} + (\dot{\vec{r}} \cdot \vec{\nabla})\vec{A}, \quad (6)$$

and $\frac{\partial \vec{A}(\vec{r})}{\partial t} = 0$ here.

c) Using the forms of \vec{A} and V that you obtained in a), check the above equation of motion is the same as the Newton's second law in the rotating frame.

Problem 2. Normal modes of a string: a different boundary condition (10 points for from *a*) to *d*).

A string with one end fixed at $x = 0$ as $u(x = 0, t) = 0$. The other end attached to a ring with the mass M which moves frictionlessly along a rod located at $x = L$. The tension along the string is T and the linear density of the string is ρ . We assume there is no gravity. The wave equation has been derived in class as

$$\frac{\partial^2}{\partial t^2} u(x, t) = c^2 \frac{\partial^2}{\partial x^2} u(x, t) \quad (7)$$

with $c^2 = T/\rho$. But the boundary condition at $x = L$ is different.

a) By performing the force analysis on the ring along the rod direction, derive the boundary condition at $x = L$.

Define a dimensionless parameter $\gamma = \rho L/M$, which is the ratio between the string mass and the ring mass. γ is convenient to use for the following questions.

b) Find the equation to determine the eigen-wavevector k_n where n is the index for the eigen-modes. Please specify what values n can take.

c) Solve the eigen-wavevector k_n and the corresponding normal modes at $\gamma = +\infty$, or, $M = 0$.

d) Solve the eigen-wavevector k_n and the corresponding normal modes at $\gamma = 0$, or, $M = +\infty$.

Parts *e*) and *f*) are for extra credit (5 points)

e) Solve the eigen-wavevectors k_n as $\gamma \rightarrow +\infty$ to the leading order of $1/\gamma$.

f) Solve the eigen-wavevectors k_n as $\gamma \rightarrow 0$ to the leading order of γ .

Problem 3. Strain and stress (5 points)

Consider an isotropic solid with the following displacement field

$$u_x(x, y, z) = \tau(z y + x^2), \quad u_y(x, y, z) = \tau(z x + y^2), \quad u_z(x, y, z) = \tau(x y + z^2), \quad (8)$$

where τ is a coefficient. The shear modulus and bulk modulus are denoted as SM and BM , respectively.

- a) Calculate the strain-tensor field E .
- b) Calculate the stress-tensor field Σ , and express it in terms of SM and BM .

Problem 4. Relativistic transformation of electric and magnetic fields (10 points)

Let us consider an infinitely large capacitor at rest in the frame S with two plates parallel to the xy -plane. The surface charge densities are $\pm\sigma$ on the upper and lower plates, respectively.

a) What are \vec{E} and \vec{B} in the frame S ?

b) Consider another frame S' with a relative velocity \vec{v} with respect to S along the x -axis. In this frame, what is the surface charge density $\pm\sigma'$? What is the current density \vec{j} ? What are \vec{E}' and \vec{B}' that you can obtain from your E&M knowledge?

c) Can you arrive at the same results as you just got from b) by applying the Lorentz transformation on the \vec{E} and \vec{B} that you got in a)?