Runge - Lenz vecter

① $\vec{A} = \frac{1}{\mu\gamma}\vec{p}\times\vec{l} - \hat{r}$, where $\hat{r} = \vec{r}/r$

$$\frac{d}{dt}(\vec{p}\times\vec{l}) = \dot{\vec{p}}\times\vec{l} + \vec{p}\times\dot{\vec{l}} = -\frac{\gamma}{r^3}\vec{r}\times\vec{l}$$

$$= -\frac{\gamma}{r^3}\vec{r}\times(\vec{r}\times\vec{p}) = \frac{\gamma}{r^3}\left[r^2\vec{p} - (\vec{r}\cdot\vec{p})\vec{r}\right]$$

$$= \frac{\mu\gamma}{r^3}\left(r^2\dot{\vec{r}} - (\vec{r}\cdot\dot{\vec{r}})\vec{r}\right)$$

$$\frac{d}{dt}\hat{r} = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{-1}{r^2}\frac{dr}{dt}\vec{r} + \frac{1}{r}\frac{d\vec{r}}{dt}$$

$$= \frac{1}{r^3}\left[r^2\dot{\vec{r}} - r\frac{dr}{dt}\vec{r}\right]$$

$$r\frac{dr}{dt} = \frac{1}{2}\frac{dr^2}{dt} = \frac{1}{2}\frac{d(\vec{r}\cdot\vec{r})}{dt} = \vec{r}\cdot\frac{d\vec{r}}{dt}$$

$$\Rightarrow \frac{d}{dt}(\vec{p}\times\vec{l})\cdot\frac{1}{\mu\gamma} = \frac{d}{dt}\hat{r}$$

$$\Rightarrow \boxed{\frac{d}{dt}\vec{A} = 0}$$

② $\vec{A}\cdot\vec{l} = \left(\frac{1}{\mu\gamma}\vec{p}\times\vec{l} - \hat{r}\right)\cdot\vec{l} = \frac{1}{\mu\gamma}(\vec{p}\times\vec{l})\cdot\vec{l} - \hat{r}\cdot(\vec{r}\times\vec{p})$

$$= 0$$

③ Calculate A^2

$$A^2 = \left(\frac{1}{\mu\gamma}\vec{p}\times\vec{l} - \hat{r}\right)^2 = \frac{1}{(\mu\gamma)^2}(\vec{p}\times\vec{l})\cdot(\vec{p}\times\vec{l}) - \frac{2}{\mu\gamma}(\vec{p}\times\vec{l})\cdot\hat{e}_r + 1$$

$$(\vec{p}\times\vec{l})\cdot(\vec{p}\times\vec{l}) = p^2 l^2 - (\vec{p}\cdot\vec{l})^2 = p^2 l^2$$

$$(\vec{p}\times\vec{l})\cdot\hat{e}_r = \frac{\vec{r}\cdot(\vec{p}\times\vec{l})}{r} = \frac{\vec{l}\cdot(\vec{r}\times\vec{p})}{r} = \frac{l^2}{r}$$

$$\Rightarrow A^2 = \frac{1}{(\mu\gamma)^2}p^2 l^2 - \frac{2}{\mu\gamma}\frac{l^2}{r} + 1 = \frac{2}{\mu\gamma^2}\left(\frac{p^2}{2\mu} - \frac{\gamma}{r}\right)l^2 + 1$$

$$= \frac{2El^2}{\mu\gamma^2} + 1$$

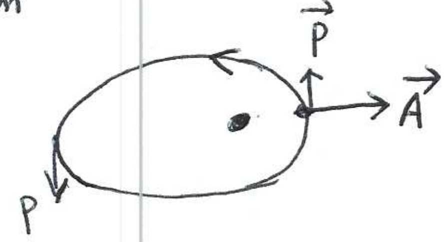
From the relation $E = \frac{\gamma^2\mu}{2l^2}(e^2-1) \Rightarrow \frac{2E}{\mu\gamma^2}+1 = e^2$

$\Rightarrow |A| = e.$

④ direction of $\vec{A}$. since $\vec{A}\perp\vec{l}$, $\vec{A}$ lies in the orbit plane.

$\vec{A}$ is conserved, we can determin $\vec{A}$'s direction

at the perihelion, At this case



$\vec{p}\times\vec{l}$ and $\hat{r}$ are colinear.

$\Rightarrow$ $\vec{A}$ is along major axis.

(At perihelion, p is large, thus $\vec{A}$ along from the center to the perigee.