Lect 6 - energy, work (I)

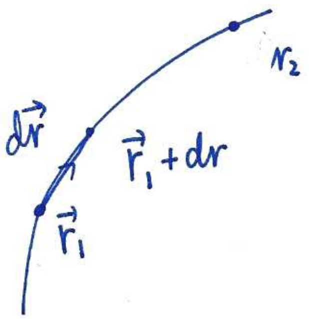
We often seek conserved quantities, or, constants of motion, to reduce the Newton's 2nd law to 1st order differential equation.

$$\vec{F} = m\frac{d^2\vec{r}}{dt^2} \Rightarrow \vec{F}\cdot d\vec{r} = m\, d\vec{r}\cdot\frac{d^2\vec{r}}{dt^2} = d\left[\frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2\right]$$

or $\boxed{dT = \vec{F}\cdot d\vec{r}}$ where $T = \frac{1}{2}mv^2$ - kinetic energy

$\vec{F}\cdot d\vec{r}$ - work

work - kinetic energy theorem.



$$T_2 - T_1 = \int_1^2 \vec{F}\cdot d\vec{r}$$

along the path from $1 \to 2$

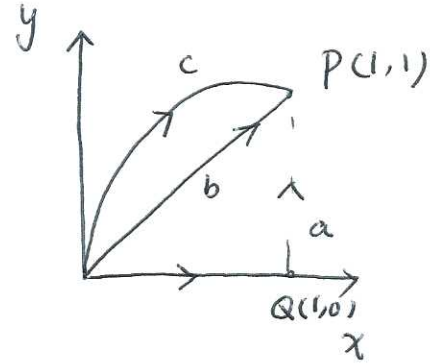
line integral

Question 1: For a general force $\vec{F}$, does $\int_1^2 \vec{F}\cdot d\vec{r}$ only depend on the initial and ending points 1 and 2, or, it also depends the concrete paths?

—— whether $\vec{F}\cdot d\vec{r}$ can be written as a total derivative?

if so, we can express $\vec{F}\cdot d\vec{r} = -dU$.

Example: $\vec{F} = y\,\hat{x} + 2x\,\hat{y}$



. Calculate $\int_O^P \vec{F}\cdot d\vec{r}$ along paths 1, 2, 3

Solution:

① $\int_0^P d\vec{r}\cdot\vec{F} = \int_0^Q dx\,F_x + \int_Q^P dy\,F_y = \int_0^1 dx\cdot 0 + \int_0^1 dy\cdot 2 = 2$

② $\int_0^P d\vec{r}\cdot\vec{F} = \int dx\,F_x + dy\,F_y = \int_0^1 dx\cdot y + \int_0^1 dy\cdot 2x = \int_0^1 x\,dx + \int_0^1 dy\cdot 2y$

$= \frac{1}{2} + 1 = 1.5$

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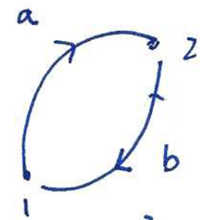
③ $x = 1-\cos\theta$ $\Rightarrow$ $d\vec{r}\cdot\vec{F} = dx\cdot y + dy\cdot 2x$

$y = \sin\theta$ $= [\sin^2\theta + 2(1-\cos\theta)\cos\theta]\,d\theta$

$$W = \int_0^{\pi/2} \left[-\frac{1}{2} - \frac{3}{2}\cos 2\theta + 2\cos\theta\right] d\theta = -\pi/4 + 2 \approx 1.21$$

So, not every force $\vec{F}(\vec{r})$ can result in $\int_1^2 \vec{F}\cdot d\vec{r}$ independent of the paths conneting 1 and 2. For those forces satisfying this condition, they are denoted as <u>conservative forces</u>. Conservative forces can also be defined as: Along any closed paths, the work done $\boxed{\oint \vec{F}\cdot d\vec{r} = 0.}$

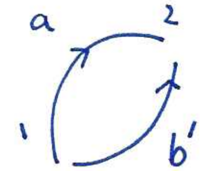
proof: ① if $\vec{F}$ is conservative, then consider a closed loop from $1 \xrightarrow{a} 2 \xrightarrow{b} 1$. Then



$$\oint \vec{F}\cdot d\vec{r} = \underbrace{\int_1^2 d\vec{r}\cdot\vec{F}}_{\text{through "a"}} + \underbrace{\int_2^1 d\vec{r}\cdot\vec{F}}_{\text{through } b} = \underbrace{\int_1^2 d\vec{r}\cdot\vec{F}}_{a} - \underbrace{\int_1^2 d\vec{r}\cdot\vec{F}}_{\text{reverse } b}$$

$$= 0.$$

② prove: If for any loop $\oint \vec{F}\cdot d\vec{r} = 0 \Rightarrow \int_1^2 d\vec{r}\cdot\vec{F}$ is independent of paths conneting 1 and 2.



Consider two paths $\underset{a}{\int_1^2} d\vec{r}\cdot\vec{F}$ and $\underset{b'}{\int_1^2} d\vec{r}\cdot\vec{F}$.

revese the direction of b', then $1 \xrightarrow{a} 2 \xrightarrow{-b'} 1$, form a closed loop

$$\oint d\vec{r}\cdot\vec{F} = 0 \Rightarrow \underset{a}{\int_1^2} d\vec{r}\cdot\vec{F} + \underset{-b'}{\int_2^1} d\vec{r}\cdot\vec{F} = 0$$

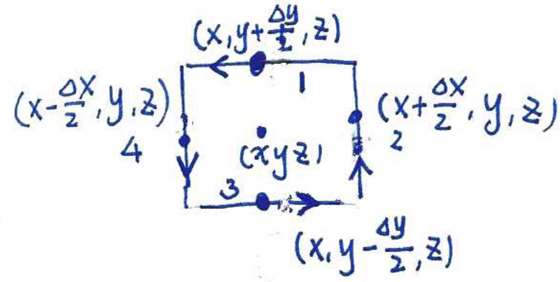
$$\Rightarrow {}_a\int_1^2 d\vec{r}\cdot\vec{F} = -\underset{\text{"}-b'\text{"}}{\int_2^1} d\vec{r}\cdot\vec{F} = \underset{b}{\int_1^2} d\vec{r}\cdot\vec{F}.$$

§: $\boxed{\text{Curl free}} \longleftrightarrow$ Conservative forces.

$\oint \vec{F}\cdot d\vec{r} = 0$ is the integral form for the condition of conservative forces. Now we will derive a differential form for this condition

Consider a small loop in the xy-plane around point $\vec{r} = (x\,y\,z)$

$$\oint \vec{F}\cdot d\vec{r} = \int_1 dx\, F_x + \int_2 dy\, F_y + \int_3 dx\, F_x + \int_4 dy\, F_y$$



$$\int_1 + \int_3 = -\Delta x \cdot \left[F_x(x, y+\tfrac{\Delta y}{2}, z) - F_x(x, y-\tfrac{\Delta y}{2}, z) \right]$$

$$= -\Delta x \frac{\partial F_x}{\partial y} \Delta y$$

Similarly $\int_2 + \int_4 = +\Delta y\, F_y(x+\frac{\Delta x}{2}, y, z) - \Delta y\, F_y(x-\frac{\Delta x}{2}, y, z)$

$$= +\Delta x \Delta y \frac{\partial F_y}{\partial x}$$

$$\Rightarrow \oint \vec{F}\cdot d\vec{r} = \Delta x \Delta y \left[-\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x} \right] = 0 \Rightarrow \left(-\frac{\partial F_x}{\partial y} + \frac{\partial F_y}{\partial x} \right) = 0$$

Similarly for loops in the yz and zx-planes, we have

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0, \quad \frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} = 0.$$

define
$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial}{\partial z} F_x - \frac{\partial}{\partial x} F_z \\ \frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \end{pmatrix}.$$

$$\Rightarrow \boxed{\nabla \times \vec{F}(\vec{r}) = 0 \Rightarrow \text{conservative force}}$$

Example of conservative forces

① $\vec{G} = -mg\hat{z}$, $\Rightarrow$ $\nabla\times\vec{G} = 0$

② $\vec{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$, we also have $\nabla\times\vec{F} = 0$.

In fact, for any central force, $\vec{F} = f(r^2)\vec{r}$, we can prove $\nabla\times\vec{F} = 0$

$$\vec{F} = f(x^2+y^2+z^2)(x\hat{x} + y\hat{y} + z\hat{z})$$

$$(\nabla\times\vec{F})_z = \frac{\partial}{\partial x}[f(r^2)y] - \frac{\partial}{\partial y}[f(r^2)x] = y\frac{df(r^2)}{d(r^2)}\frac{dr^2}{dx} - x\frac{df(r^2)}{d(r^2)}\cdot\frac{dr^2}{dy}$$

$$= \frac{df(r^2)}{d(r^2)}[2xy - 2xy] = 0$$, $\frac{df(r^2)}{d(r^2)}$: ~~treat~~ r^2 as a single variable

Similarly we have $(\nabla\times\vec{F})_x = (\nabla\times\vec{F})_y = 0$.

$\boxed{\text{All central forces are conservative}}$: Gravity, electro-static force

§: A mathematical statement:

Any conservative force $\vec{F}$, i.e $\vec{F}$ satisfying $\nabla\times\vec{F} = 0$,

can be expressed as the gradient of a scalar function as

$\vec{F} = -\nabla U(\vec{r})$, where $U(\vec{r})$ is called potential

(scalar). $\nabla U(\vec{r})$ is defined as $\boxed{\frac{\partial}{\partial x}U(\vec{r})\hat{x} + \frac{\partial}{\partial y}U(\vec{r})\hat{y} + \frac{\partial}{\partial z}U(\vec{r})\hat{z} = \nabla U(\vec{r})}$

or $\boxed{\vec{F}_i = -\dfrac{\partial}{\partial x_i} \mathcal{U}(\vec{r})}$ for $i = 1, 2, 3$. (we also use (x_1, x_2, x_3) as (x, y, z)).

Why $\mathcal{U}(\vec{r})$ is useful? — for motions in the conservative force field, we have

$$\frac{d}{dt}[T] = \vec{F} \cdot \frac{d\vec{r}}{dt},$$

$$\left\{ d\,\mathcal{U}(\vec{r}) = \frac{\partial \mathcal{U}}{\partial x} dx + \frac{\partial \mathcal{U}}{\partial y} dy + \frac{\partial \mathcal{U}}{\partial z} dz \right.$$

$$= \nabla \mathcal{U} \cdot d\vec{r} = -\vec{F} \cdot d\vec{r}$$

$$\Rightarrow \frac{d}{dt}[T + \mathcal{U}] = 0, \quad \text{or,} \quad \boxed{T + \mathcal{U} = E}.$$

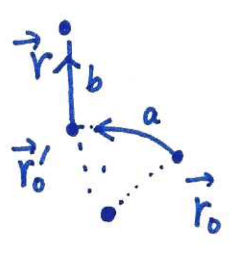
- Example

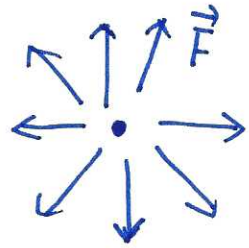
① $\vec{G} = -mg\hat{z}$

$$\mathcal{U}(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{G} \cdot d\vec{r} = mg(z - z_0) \rightarrow mgz \quad \text{(drop a constant)}.$$

② Coulumb potential $\vec{F}(r) = \dfrac{kqQ}{r^2} \hat{e}_r$

$$\mathcal{U}(\vec{r}) = -\underset{a}{\int_{\vec{r}_0}^{\vec{r}_0'}} d\vec{r} \cdot \vec{F} - \int_{\vec{r}_0'}^{\vec{r}} d\vec{r} \cdot \vec{F}$$





path a, $d\vec{r} \cdot \vec{F} = 0$

path b $d\vec{r} \cdot \vec{F} = \dfrac{kqQ}{r^2} dr \quad \Rightarrow \mathcal{U}(\vec{r}) = -\displaystyle\int_{r_0}^{r} \frac{kqQ\,dr}{r^2} = kqQ\left[\frac{1}{r} - \frac{1}{r_0}\right]$

we can drop the constant term and choose

$$\boxed{U(r) = \frac{kqQ}{r}}.$$

§ more discussions.

① several forces: if all of them are conservative, we can define potential energy for each of them: $\vec{F}_1 = -\nabla U_1$, $\vec{F}_2 = -\nabla U_2$, ...

The energy $\boxed{E = T + U = T + U_1(\vec{r}) + \cdots U_n(\vec{r})}$ is conserved.

② if some forces are non-conservative,

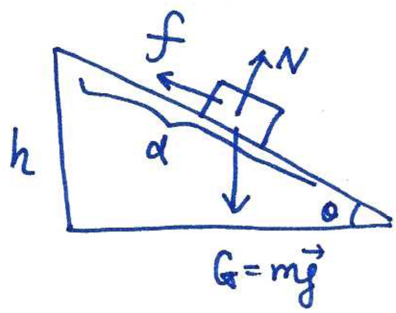
$\Delta T = W = W_{con.} + W_{nc}$, for $W_{con.}$, we have $W_{con} = -\Delta U$

$$\Rightarrow \Delta(T+U) = W_{nc}.$$

A typical non-conservative force is friction, which causes dissipation

Example: block sliding down an incline

the normal force doesn't do work.



$U = mgy$

$\Rightarrow \quad \Delta(T+U) = -fd = -mg\cos\theta\,\mu \cdot d$

$T_{in} = 0$, $U_{in} = mgh = mgd\sin\theta$

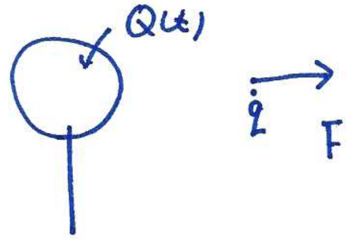
$T_f = ?$ $\quad U_f = 0$

$$\Rightarrow T_f = mgd\sin\theta - mg\cos\theta\,\mu d = \frac{1}{2}mv_f^2 \Rightarrow v_f = \sqrt{2gd(\sin\theta - \mu\cos\theta)}$$

§ Time-dependent potential energy

if $\vec{F}(\vec{r},t)$ satisfies $\nabla\times\vec{F}(\vec{r},t)=0$, but it's time-dependent, then we can still write $\vec{F}(\vec{r},t) = -\nabla\mathcal{U}(\vec{r},t)$. Nevertheless $E = T+\mathcal{U}$ is no-longer conserved.

For a changing charge $Q(t)$, we can still define $\mathcal{U}(\vec{r},t) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,',t)\,d\vec{r}\,'$.



$$\vec{F} = \frac{kqQ(t)}{r^2}\hat{r}$$

Now check $dT = \dfrac{dT}{dt}\,dt = \dfrac{d}{dt}\left(\tfrac{1}{2}mv^2\right)dt = m\dot{\vec{v}}\cdot\vec{v}\,dt = \vec{F}\cdot d\vec{r}$

$$d\mathcal{U}(\vec{r},t) = \frac{\partial\mathcal{U}}{\partial x}dx + \frac{\partial\mathcal{U}}{\partial y}dy + \frac{\partial\mathcal{U}}{\partial z}dz + \frac{\partial\mathcal{U}}{\partial t}dt$$

$$= \nabla\mathcal{U}\cdot d\vec{r} + \frac{\partial\mathcal{U}}{\partial t}dt = -\vec{F}\cdot d\vec{r} + \frac{\partial\mathcal{U}}{\partial t}dt$$

$$\Rightarrow dT = -d\mathcal{U} + \frac{\partial\mathcal{U}}{\partial t}dt \qquad \Rightarrow \boxed{d(T+\mathcal{U}) = \frac{\partial\mathcal{U}}{\partial t}dt}$$