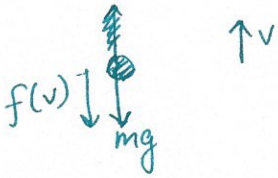
Taylor 2.41 :



For the upward motion of the ball, from Newton's II law

$$m\dot{v} = -mg - f(v)$$

$$\Rightarrow \quad \frac{dv}{dt} = -g - \frac{c}{m}v^2$$

~~Set~~ $\cancel{\frac{dv}{dt} = 0}$ ~~to get~~ $\cancel{v_{ter}^2 = \frac{mg}{c}}$

Define $v_{ter} = \sqrt{\frac{mg}{c}}$ to get

$$\frac{dv}{dt} = -g\left[1 + \frac{v^2}{v_{ter}^2}\right]$$

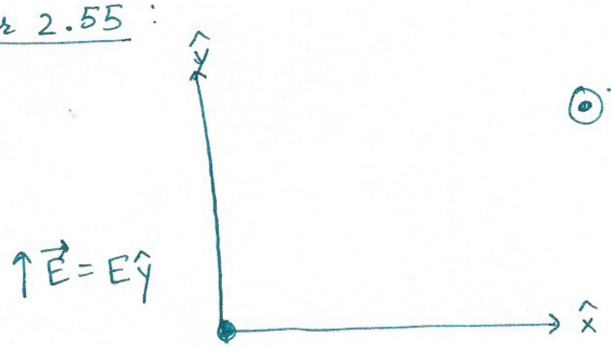
$$\Rightarrow \quad v\frac{dv}{dy} = -g\left(1 + \frac{v^2}{v_{ter}^2}\right)$$

$$\Rightarrow \quad \int_{v_0}^{v} \frac{d(v^2)}{1 + v^2/v_{ter}^2} = -2g\int_0^y dy$$

$$v_{ter}^2\left(\ln\left(1 + \frac{v^2}{v_{ter}^2}\right)\Bigg|_{v_0}^{v}\right) = -2gy$$

$$\Rightarrow \quad \boxed{v_{ter}^2 \ln\left(\frac{1 + v^2/v_{ter}^2}{1 + v_0^2/v_{ter}^2}\right) = -2gy}$$

Taylor 2.55 :



(a) Using Newton's II law & the Lorentz force law,

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) \quad (1)$$

let $\vec{v}(t) = v_x(t)\,\hat{x} + v_y(t)\,\hat{y} + v_z(t)\,\hat{z}$

Plugging this into the equation, we get

$$m\frac{dv_x}{dt} = q\left(v_y B\right) \Rightarrow \dot{v}_x = \frac{qB}{m}v_y \quad (2)$$

$$m\frac{dv_y}{dt} = q\left(E - v_x B\right) \Rightarrow \dot{v}_y = \frac{qE}{m} - \frac{qB}{m}v_x \quad (3)$$

$$m\frac{dv_z}{dt} = q(0) \Rightarrow \dot{v}_z = 0 \quad (4)$$

~~$\because$ the $\dot{v}_z = 0$~~

$\because$ $\dot{v}_z = 0$ & the initial z velocity is zero, $v_z(t) = 0$. $\Rightarrow$ The motion stays confined to the $z=0$ plane.

(b) setting $\dot{v}_y = 0$, we get $v_x = \frac{E}{B}$.

If $v_x = E/B$, $\dot{v}_y = 0 \Rightarrow v_y =$ constant.

$\because$ the initial y velocity is zero, $\Rightarrow v_y = 0. \Rightarrow \dot{v}_x = 0$ ~~$\Rightarrow v_x = 0$~~

(from (2))

~~So, the parti~~

$\Rightarrow v_x =$ constant.

$\Rightarrow$ Particle moves undeflected through the fields.

(c) As suggested in the problem, change variables to

$$u_x = V_x - E/B$$

$$u_y = V_y$$

Then the equation become

$$\dot{u}_x = \frac{qB}{m} u_y$$

$$\dot{u}_y = \frac{qE}{m} - \frac{qB}{m}\left(u_x + E/B\right) = -\frac{qB}{m} u_x$$

$$\dot{u}_y = -\frac{qB}{m} u_x$$

Comparing with 2.68, we get

$$u_x = A_1 \cos\omega t + A_2 \sin\omega t, \quad \cancel{\omega = \sqrt{\frac{qB}{m}}} \quad \boxed{\omega = \frac{qB}{m}}$$

$$u_y = -\frac{A_1}{\omega} \sin\omega t + \frac{A_2}{\omega} \cos\omega t$$

$$\Rightarrow V_x(t) = E/B + A_1 \cos\omega t + A_2 \sin\omega t$$

$$V_y(t) = -A_1/\omega \sin\omega t + A_2/\omega \cos\omega t$$

Given $V_x(t=0) = V_{x0}$

$V_y(t=0) = 0,$

$$V_{x0} = E/B + A_1$$

$$0 = A_2$$

$$\Rightarrow \boxed{\begin{aligned} V_x(t) &= E/B\left(1 - \cos\omega t\right) + V_{x0}\cos\omega t \\ V_y(t) &= \left(E/B - V_{x0}\right)\sin\omega t \end{aligned}}$$

(d) By definition,

$$\frac{dx}{dt} = v_x$$

$$\frac{dy}{dt} = v_y$$

$$\Rightarrow \quad \frac{dx}{dt} = E/B\,(1-\cos\omega t) + v_{x0}\cos\omega t$$

$$\frac{dy}{dt} = \left(\frac{E}{B} - v_{x0}\right)\sin\omega t$$

Assuming the particle starts at the origin,

$$\int_0^x dx = \int_0^t \left[E/B(1-\cos\omega t) + v_{x0}\cos\omega t\right] dt$$

$$\Rightarrow \quad \boxed{x(t) = \frac{E}{B}t + \frac{1}{\omega}\left(v_{x0} - E/B\right)\sin\omega t}$$

$$\int_0^y dy = \int_0^t \left(\frac{E}{B} - v_{x0}\right)\sin\omega t\, dt$$

$$\boxed{y(t) = \frac{1}{\omega}\left(v_{x0} - E/B\right)(\cos\omega t - 1)}$$

N.B : $$\left(x - \frac{E}{B}t\right)^2 + \left(y + \frac{1}{\omega}\left(v_{x0} - \frac{E}{B}\right)\right)^2 = \frac{\left(v_{x0} - \frac{E}{B}\right)^2}{\omega^2}$$

$\Rightarrow$ The trajectory of the particle is a circle centred at $\left(\frac{E}{B}t, -\frac{1}{\omega}(v_{x0} - E/B)\right)$ with radius $\frac{v_{x0} - E/B}{\omega}$.

This trajectory, called a cycloid, is traced by a point on the circumference of a ring ~~of~~ radius $\frac{v_{x0} - E/B}{\omega}$ rolling along x axis with linear speed $\frac{E}{B}$ along $\hat{x}$.

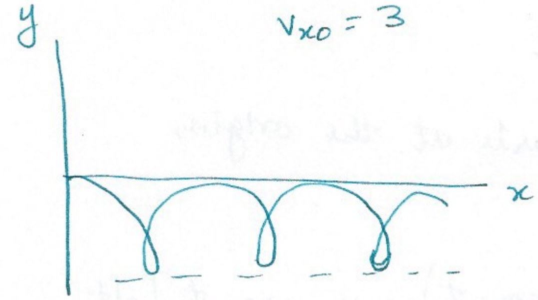
We can rescale time & velocity in the problem by setting $\frac{E}{B} = 1$

& $\omega = 1$ to get

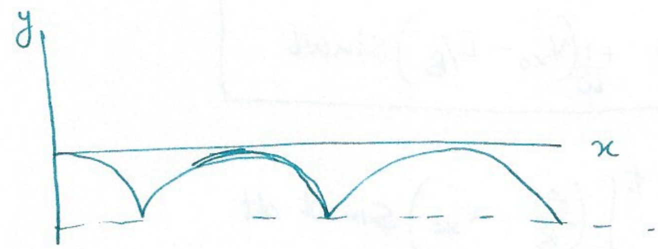
$$x(t) = t + (v_{xo} - 1)\sin t$$
$$y(t) = (v_{xo} - 1)(\cos t - 1)$$

Plotting this for different values of v_{xo}, we get :

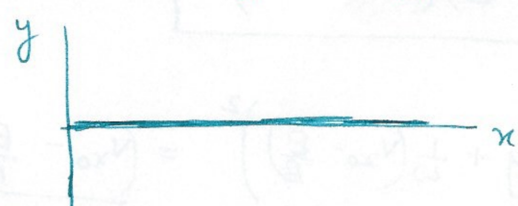
$V_{xo} = 3$



$V_{xo} = 2$



$V_{xo} = 1 = V_{dr}$

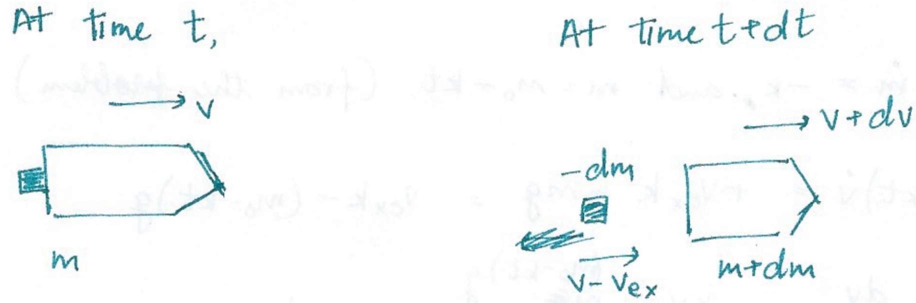


3.11

(a) Newton's II says

$$\frac{d\vec{p}}{dt} = \vec{F}, \text{ where } \vec{p} \text{ is the momentum of the system.}$$

In most problems, where the mass of the system does not change, $\frac{d\vec{p}}{dt}$ reduces to $m\frac{d\vec{v}}{dt}$. In this case however, the mass of the rocket changes as fuel is ejected out. Calculating dp then becomes slightly more complicated.



At time t, the rocket of mass m is moving with a velocity v. At time t+dt, a small mass −dm of the exhaust is ejected from the rocket at velocity $-V_{ex}$ with respect to the rocket. Then, in the "Lab frame".

$$p(t) = mv$$

$$p(t+dt) = \underbrace{(m+dm)(v+dv)}_{\text{momentum of the rocket}} + \underbrace{(-dm)(v-V_{ex})}_{\text{momentum of the exhaust}}$$

$$= mv + mdv + dmdv + V_{ex}dm$$

$$\Rightarrow dp = p(t+dt) - p(t) = mdv + V_{ex}dm + dmdv$$

To first order, we can neglect $dmdv$, to get

$$dp = mdv + V_{ex}dm \quad \Rightarrow \quad dp/dt = m\,dv/dt + V_{ex}\,dm/dt = m\dot{v} + V_{ex}\dot{m}$$

With the external force F_{ext}, we can write

$$\frac{dp}{dt} = F_{ext} \quad (\text{II law})$$

$$\Rightarrow \quad m\dot{v} + v_{ex}\dot{m} = F_{ext}$$

$$\Rightarrow \quad \boxed{m\dot{v} = -v_{ex}\dot{m} + F_{ext}}$$

(b) Lets say the upward direction is $+\hat{y}$. Then $F_{ext} = -mg\hat{y}$. Plugging this in,

$$m\dot{v} = -v_{ex}\dot{m} - mg$$

Given $\dot{m} = -k$, and $m = m_0 - kt$ (from the problem)

$$(m_0 - kt)\dot{v} = +v_{ex}k - mg = v_{ex}k - (m_0 - kt)g$$

$$\Rightarrow \quad \frac{dv}{dt} = \frac{kv_{ex} - \cancel{mg}\,(m_0-kt)g}{m_0 - kt} = \frac{kv_{ex}}{m_0 - kt} - g$$

$$\int_0^v dv = \int_0^t \frac{kv_{ex} - \cancel{mg}}{m_0 - kt}\,dt - \int_0^t g\,dt$$

given rocket takes off from rest

$$\Rightarrow \quad v = -\frac{1}{k}\left(kv_{ex} - \cancel{mg}\right)\ln\left(\frac{m_0 - kt}{m_0}\right) - gt$$

$$\Rightarrow \quad \cancel{\boxed{v(t) = \left(v_{ex} - \frac{mg}{k}\right)\ln\left(\frac{m_0}{m_0 - kt}\right)}}$$

$$\Rightarrow \quad \boxed{v = v_{ex}\ln\left(\frac{m_0}{m_0 - kt}\right) - gt}$$

(c) Say $V_{ex} = 3000 m/s$, $g = 9.8 m/s^2$, $\frac{m_0}{m_0 - kt} = 2$ & $t = 2min = 120s$

$$v(120s) = 3000 \ln(2) - (9.8)(120)$$

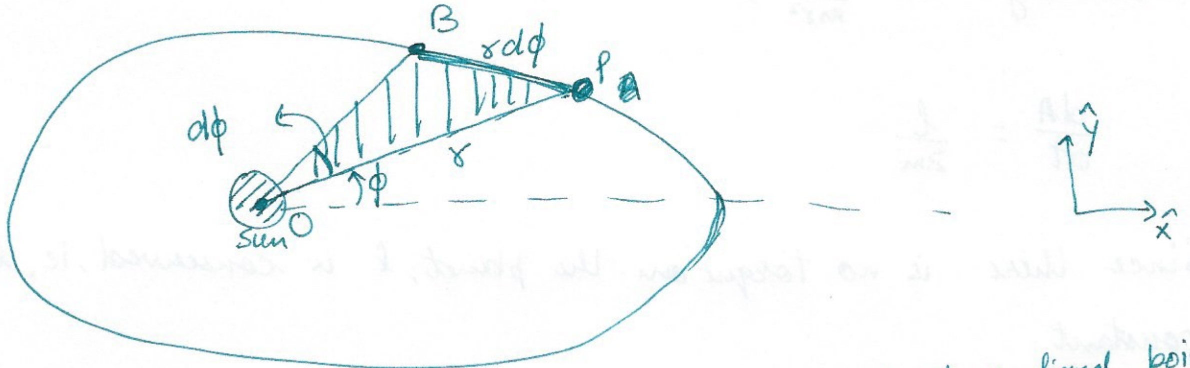
$$\approx 900 m/s$$

In the ~~absence~~ absence of gravity (ie, $g = 0$)

$$v(120s) \approx 2100 m/s$$

(d) If $k V_{ex} < mg$, the rocket will not be able to take off and just press against the ground. But as more mass is shed by the rocket "mg" decreases & eventually $k V_{ex} > mg$ and the rocket takes off.

3.27



(a) Since the motion is in a plane in an orbit about a fixed point, the problem lends itself to treatment in polar coordinates.

Now,

$\vec{l} = \vec{r} \times \vec{p}$ (by definition)

$\vec{p} = m\vec{v}$

$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$ (in polar coordinates)

$\vec{r} = r\hat{r}$ (in polar coordinates)

$$\Rightarrow \vec{l} = (r\hat{r}) \times m(\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$

$$= mr^2\dot{\phi}\hat{z} \qquad (\hat{r} \times \hat{\phi} = \hat{z})$$

$$\Rightarrow l = mr^2\dot{\phi} = mr^2\omega \qquad (\omega \equiv \dot{\phi})$$

(b) Consider the ~~planet~~ planet to be at point P at time t, at a distance r from the sun at angle ϕ with the x axis. In a small time dt, the planet reaches B, sweeping an angle $d\phi$. If dt is small enough, we can treat ΔOPB as a right triangle with $OP = r$ & $PB = rd\phi$. The area swept in time dt therefore is,

$$dA = \frac{1}{2} \times OP \times PB = \frac{1}{2} r(rd\phi) = \frac{1}{2} r^2 d\phi$$

Dividing both sides by dt,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\phi}{dt} = \frac{r^2}{2}\omega$$

Substituting $\omega = \frac{l}{mr^2}$,

$$\frac{dA}{dt} = \frac{l}{2m}$$

Since there is no torque on the planet, l is conserved, ie, l is constant.

$$\Rightarrow \frac{dA}{dt} = \text{constant.}$$

$\Rightarrow$ The ~~planet~~ planet sweeps out equal areas in equal times.