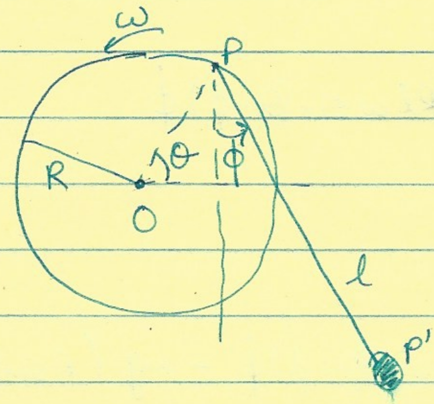
PHYS 110A HW # 8

Taylor 7.29



Say ~~the point~~ OP makes an angle $\theta(t)$ with the x axis &

& PP' makes an angle ϕ with the y axis

The the coordinates of P are $(R\cos\theta, R\sin\theta)$

$\Rightarrow$ coordinates of P' are $x = R\cos\theta + l\sin\phi$, $y = R\sin\theta - l\cos\phi$

Given the wheel rotates with ~~avg~~ angular velocity ω,

$\theta(t) = \omega t + \theta_0$ (assuming the wheel starts $\theta = \theta_0$)

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$\dot{x} = -R\sin\theta\dot{\theta} + l\cos\phi\dot{\phi}$$

$$\dot{y} = R\cos\theta\dot{\theta} + l\sin\phi\dot{\phi}$$

$$\Rightarrow T = \frac{1}{2} m \left(R^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2Rl\sin\theta\cos\phi\dot{\phi}\dot{\theta} + 2lR\cos\theta\sin\phi\dot{\theta}\dot{\phi} \right)$$

$$= \frac{1}{2} m \left(R^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2Rl\sin(\theta-\phi)\dot{\theta}\dot{\phi} \right)$$

$$\dot{\theta} = \omega$$

$$\Rightarrow T = \frac{1}{2} m R^2\omega^2 + \frac{1}{2} m l^2\dot{\phi}^2 - Rl\omega\sin(\theta-\phi)\dot{\phi}$$

The Lagrangian is

$$\mathcal{L} = T - U$$

$$T = \frac{1}{2} m R^2 \omega^2 + \frac{1}{2} m l^2 \dot{\phi}^2 - Rl\omega \sin(\theta - \phi)\, \dot{\phi}$$

$$U = +mg\,(R\sin\theta - l\cos\phi)$$

$$\Rightarrow \mathcal{L} = \frac{1}{2} m R^2 \omega^2 + \frac{1}{2} m l^2 \dot{\phi}^2 - Rl\omega \sin(\theta - \phi)\dot{\phi} - mg\,(R\sin\theta - l\cos\phi)$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\Rightarrow \frac{d}{dt}\left(ml^2\dot{\phi} - Rl\omega \sin(\theta - \phi)\right) = Rl\omega \cos(\theta - \phi)\dot{\phi} - mgl\sin\phi$$

$$\Rightarrow ml^2\ddot{\phi} - Rl\omega\cos(\theta - \phi)(\dot{\theta} - \dot{\phi}) = Rl\omega\cos(\theta - \phi)\dot{\phi} - mgl\sin\phi$$

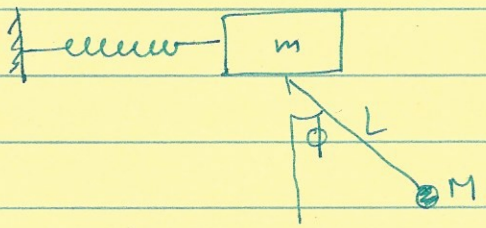
$$\Rightarrow ml^2\ddot{\phi} - Rl\omega\dot{\theta}\cos(\theta - \phi) = -mgl\sin\phi$$

$$\Rightarrow \boxed{ml^2\ddot{\phi} = Rl\omega^2\cos(\theta - \phi) - mgl\sin\phi}$$

For $\omega = 0$,

$ml^2\ddot{\phi} = -mgl\sin\phi$ which is the equation of motion for a simple pendulum.

Taylor 7.31



The coordinates of the cart are $(x, 0)$

The coordinates of the pendulum are $(x + L\sin\phi, -L\cos\phi)$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} M\left[(\dot{x} + L\cos\phi\,\dot{\phi})^2 + (L\sin\phi\,\dot{\phi})^2 \right]$$

$$= \frac{1}{2} m\dot{x}^2 + \frac{1}{2} M\left[\dot{x}^2 + 2\dot{x}\dot{\phi}L\cos\phi + L^2\dot{\phi}^2 \right]$$

$$= \frac{1}{2}(m+M)\dot{x}^2 + M\dot{x}\dot{\phi}L\cos\phi + \frac{1}{2}ML^2\dot{\phi}^2$$

$$U = \frac{1}{2}kx^2 - MgL\cos\phi$$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}(m+M)\dot{x}^2 + M\dot{x}\dot{\phi}L\cos\phi + \frac{1}{2}ML^2\dot{\phi}^2 - \frac{1}{2}kx^2 + MgL\cos\phi$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \quad \text{and} \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

So,

$$\frac{d}{dt}\left((m+M)\dot{x} + M\dot{\phi}L\cos\phi \right) = -kx$$

$$\Rightarrow \boxed{(M+m)\ddot{x} + ML\ddot{\phi}\cos\phi - ML\dot{\phi}^2\sin\phi = -kx}$$

$$\frac{d}{dt}\left(M\dot{x}L\cos\phi + ML^2\dot{\phi} \right) = -M\dot{x}\dot{\phi}L\sin\phi - MgL\sin\phi$$

$$\Rightarrow ML\ddot{x}\cos\phi \cancel{- ML\dot{x}\dot{\phi}\sin\phi} + ML^2\ddot{\phi} = \cancel{-M\dot{x}\dot{\phi}L\sin\phi} - MgL\sin\phi$$

$$\Rightarrow \boxed{ML\ddot{x}\cos\phi + ML^2\ddot{\phi} = -MgL\sin\phi}$$

For ϕ & x small, $\cos\phi \approx 1$, $\sin\phi \approx \phi$, $\phi\dot{\phi}^2 << \ddot{\phi}$

$$\boxed{\begin{array}{l} (M+m)\ddot{x} + ML\ddot{\phi} \cancel{- ML\dot{\phi}^2\phi} = -kx \\ ML\ddot{x} + ML^2\ddot{\phi} = -MgL\phi \end{array}}$$

Taylor 7.41

Say the position vector of the bead are

$$\vec{r} = \rho\hat{\rho} + z\hat{z}$$

Then $\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$

Now $\dot{\phi} = \omega$

$$\Rightarrow \quad \vec{v} = \dot{\rho}\hat{\rho} + \rho\omega\hat{\phi} + \dot{z}\hat{z}$$

Given $z = k\rho^2 \quad \Rightarrow \quad \dot{z} = 2k\rho\dot{\rho}$

$$\Rightarrow \quad \vec{v} = \dot{\rho}\hat{\rho} + \rho\omega\hat{\phi} + 2k\rho\dot{\rho}\,\hat{z}$$

$$T = \tfrac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\omega^2 + 4k^2\rho^2\dot{\rho}^2\right)$$

$$U = mgz = kmg\rho^2$$

$$\mathcal{L} = T - U = \frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\omega^2 + 2k^2m\rho^2\dot{\rho}^2 - kmg\rho^2$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\rho}} = m\dot{\rho} + 4k^2 m\rho^2\dot{\rho}$$

$$\frac{\partial \mathcal{L}}{\partial \rho} = m\rho\omega^2 + 4k^2 m\rho\dot{\rho}^2 - 2kmg\rho$$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) = \frac{\partial \mathcal{L}}{\partial \rho}$$

$$\cancel{m}\ddot{\rho} + 4k^2\cancel{m}\rho^2\ddot{\rho} + 8k^2\cancel{m}\rho\dot{\rho}^2 = \cancel{m}\rho\omega^2 + 4k^2\cancel{m}\rho\dot{\rho}^2 - 2\cancel{m}gk\rho$$

$$\ddot{\rho} + 4k^2\rho^2\ddot{\rho} + 4k^2\rho\dot{\rho}^2 = -2gk\rho + \rho\omega^2$$

$$\ddot{\rho} + \cancel{4k^2\rho^2\ddot{\rho}}\; 4k^2\rho\left(\rho\ddot{\rho} + \dot{\rho}^2\right) = \left(-2gk + \omega^2\right)\rho$$

So, the equation of motion is

$$\boxed{\ddot{\rho} + 4k^2\rho\left(\ddot{\rho}\rho + \dot{\rho}^2\right) = \rho\left(\omega^2 - 2gk\right)}$$

At equilibrium, $\ddot{\rho} = \dot{\rho} = 0$.

$\Rightarrow \rho(\omega^2 - 2gk) = 0$

so $\rho = 0$ or $\omega^2 = 2gk$

So $\rho = 0$ is an equilibrium point

or if $\omega^2 = 2gk$, then the bead is at equilibrium for all values of ρ.

For $\rho = 0$, consider a small deviation $\delta\rho$ from $\rho_0 = 0$. Plugging that in equation motion upto first order in $\delta\rho$

$$\delta\ddot{\rho} = \delta\rho\,(\omega^2 - 2gk)$$

$$\Rightarrow \delta\rho = Ae^{\pm\sqrt{\omega^2 - 2gk}\,t}$$

For $\omega^2 < 2gk$, $\sqrt{\omega^2 - 2gk}$ is imaginary $\Rightarrow$ $\delta\rho$ oscillates

$\Rightarrow$ equilibrium is stable.

For $\omega^2 > 2gk$, $\sqrt{\omega^2 - 2gk}$ is real $\Rightarrow$ $\delta\rho$ grows exponentially

$\Rightarrow$ ~~$\delta\rho$~~ equilibrium is unstable

For $\omega^2 = 2gk$,

$$\ddot{\rho} + 4k^2\rho\left(\ddot{\rho}\rho + \dot{\rho}^2\right) = 0$$

$$\Rightarrow \ddot{\rho} + 4k^2\rho\left(\dot{\rho}\rho\right)^{\cdot} = 0$$

To first order $\ddot{\rho} = 0 \Rightarrow \rho = \rho_0 + vt$.

$\Rightarrow$ Equilibrium is unstable for $\omega^2 = 2gk$

7.46. The lagrangian of the system can be written with the ~~cylindrical~~ spherical polar coordinates of each particle as the generalised coordinates. Then

$$\mathcal{L} = \mathcal{L}(r_1, \theta_1, \phi_1, r_2, \theta_2, \phi_2, r_3, \theta_3, \phi_3 \ldots\ldots, r_N, \theta_N, \phi_N)$$

Given that $\mathcal{L}$ is invariant under rotations about the z axis, if each particle is displaced by a small angle ϵ about the z axis, $\mathcal{L}$ does not change.

$$\Rightarrow \mathcal{L}(r_1, \theta_1, \phi_1; r_2, \theta_2, \phi_2 \ldots, r_N, \theta_N, \phi_N)$$
$$= \mathcal{L}(r_1, \theta_1, \phi_1 + \epsilon, r_2, \theta_2, \phi_2 + \epsilon, \ldots\ldots r_N, \theta_N, \phi_N + \epsilon)$$

For small ϵ,

$$\mathcal{L}(r_1, \theta_1, \phi_1 + \epsilon, r_2, \theta_2, \phi_2 + \epsilon, \ldots, r_N, \theta_N, \phi_N + \epsilon)$$
$$\approx \mathcal{L}(r_1, \theta_1, \phi_1, r_2, \theta_2, \phi_2, \ldots r_N, \theta_N, \phi_N) + \epsilon \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha}$$

$$\Rightarrow \quad \epsilon \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0 \quad \text{for any small } \epsilon.$$

$$\Rightarrow \quad \boxed{\sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0} \qquad (1)$$

Now, the equation of motion for the ϕ_α coordinate is given by

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) = \frac{\partial \mathcal{L}}{\partial \phi_\alpha}$$

Plugging this into (1),

$$\sum_\alpha \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(\sum_\alpha \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) = 0$$

$$\Rightarrow \quad \sum_{\alpha} \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha} = \text{constant}$$

Note that $\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha} = l_{z,\alpha} =$ the z component of the angular momentum of the α-th particle

$$\Rightarrow \quad \sum_{\alpha} l_{z,\alpha} = \text{constant}$$

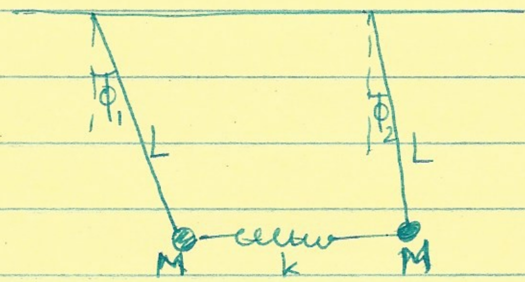
$$\Rightarrow \quad L_z = \text{constant}$$

($\because$ $L_z = \sum_{\alpha} l_{z,\alpha} =$ z component of the total angular momentum).

So, generalizing this, we can say that if the Lagrangian is invariant under rotation ~~along~~ about some axis, the component of angular momentum along that axis is conserved.

So, if $\mathcal{L}$ is invariant under rotation about all axes, ~~the~~ all components of $\vec{L}$ are conserved.

11.14



~~(a)~~ The generalised coordinate are ϕ_1 & ϕ_2.

The kinetic energies are

$$T_1 = \frac{1}{2} ML^2 \dot{\phi}_1^2 \quad , \quad T_2 = \frac{1}{2} ML^2 \dot{\phi}_2^2$$

The potential energies are

$$U_1 = -MgL\cos\phi_1 \qquad U_2 = -MgL\cos\phi_2 \qquad U_{spring} \approx \frac{1}{2}kL^2(\phi_2 - \phi_1)^2$$

$$\mathcal{L} = T_1 + T_2 - U_1 - U_2 - U_{spring}$$

$$= \frac{1}{2} ML^2(\dot{\phi}_1^2 + \dot{\phi}_2^2) + MgL(\cos\phi_1 + \cos\phi_2) - \frac{1}{2}kL^2(\phi_2 - \phi_1)^2$$

The equations of motion are:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_1}\right) = \frac{\partial \mathcal{L}}{\partial \phi_1} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_2}\right) = \frac{\partial \mathcal{L}}{\partial \phi_2}$$

$$\Rightarrow \quad ML^2\ddot{\phi}_1 = -MgL\sin\phi_1 + kL^2(\phi_2 - \phi_1)$$

$$ML^2\ddot{\phi}_2 = -MgL\sin\phi_2 - kL^2(\phi_2 - \phi_1)$$

For ϕ small, $\sin\phi \approx \phi$

$$\Rightarrow \quad ML^2\ddot{\phi}_1 = -MgL\phi_1 + kL^2(\phi_2 - \phi_1) \qquad (1)$$

$$ML^2\ddot{\phi}_2 = -MgL\phi_2 - kL^2(\phi_2 - \phi_1) \qquad (2)$$

~~Treat~~

Define $\vec{X} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

Then equations ① & ② can be written as

$$\ddot{\vec{X}} = \begin{bmatrix} -g/L - k/M & k/M \\ k/M & -g/L - k/M \end{bmatrix} \vec{X}$$

The frequencies are obtained ~~by~~ from the equation

$$\det \begin{bmatrix} -g/L - k/M + \omega^2 & k/M \\ k/M & -g/L - k/M + \omega^2 \end{bmatrix} = 0$$

$$\Rightarrow \quad \left(\frac{g}{L} + \frac{k}{M} - \omega^2 \right)^2 = \left(\frac{k}{M} \right)^2$$

$$\Rightarrow \quad \frac{g}{L} + \frac{k}{M} - \omega^2 = \pm \frac{k}{M}$$

$$\Rightarrow \quad \boxed{\omega^2 = +\frac{g}{L} \quad \text{or} \quad \omega^2 = \frac{g}{L} + \frac{2k}{M}}$$

For $\omega^2 = \frac{g}{L}$:

Say the eigen vector is $\vec{V}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Then

$$-\omega^2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} g/L - k/M & k/M \\ k/M & -g/L - k/M \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \quad -\frac{g}{L} x_1 = -\frac{g}{L} x_1 - \frac{k}{M} x_1 + \frac{k}{M} x_2$$

$$\Rightarrow \quad x_1 = x_2$$

$\Rightarrow \quad \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ upto a multiplicative factor.

$\Rightarrow$ ~~ϕ_1~~ For $\omega^2 = \frac{g}{L}$,

$$\phi_1(t) = \phi_2(t) = A e^{i\omega t} + B e^{-i\omega t}$$

i.e. both pendulums are in phase

For $\omega^2 = \frac{g}{L} + \frac{2k}{M}$,

say the eigenvector is $\vec{V}_2 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Then

$$-\omega^2 \vec{V}_2 = \begin{bmatrix} -g/L - k/M & k/M \\ k/M & -g/L - k/M \end{bmatrix} \vec{V}_2$$

$$\Rightarrow \quad -\left(\frac{g}{L} + \frac{2k}{M}\right) y_1 = \left(-\frac{g}{L} - \frac{k}{M}\right) y_1 + \frac{k}{M} y_2$$

$$\Rightarrow \quad -y_1 = y_2$$

$\Rightarrow \quad \vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ upto a multiplicative factor

$\Rightarrow$ for $\omega^2 = g/L + 2k/M$,

if $\phi_1(t) = A e^{i\omega t} + B e^{-i\omega t}$, then

$\phi_2(t) = -A e^{i\omega t} - B e^{-i\omega t}$

i.e. ϕ_1 & ϕ_2 are completely out of phase.

11.19 From problem 7.31, we have the equations of motion for small oscillations are

$$(M+m)\ddot{x} + ML\ddot{\phi} = -kx$$
$$ML\ddot{x} + ML^2\ddot{\phi} = -MgL\phi$$

Given $m = M = L = g = 1$, $k = 2$, we get

$$2\ddot{x} + \ddot{\phi} = -2x$$
$$\ddot{x} + \ddot{\phi} = -\phi$$

let $\vec{X} = \begin{pmatrix} x \\ \phi \end{pmatrix}$

Then, the equation ~~of mot~~ are:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \ddot{\vec{X}} = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix} \vec{X}$$

So, the system is of the type $M\ddot{\vec{X}} = -K\vec{X}$ with

$$M = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \& \quad K = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

The normal mode frequencies are given by

$$\det(K - \omega^2 M) = 0$$
$$\Rightarrow \begin{vmatrix} 2-\omega^2(2) & -\omega^2 \\ -\omega^2 & 1-\omega^2 \end{vmatrix} = 0$$
$$\Rightarrow 2(1-\omega^2)^2 - \omega^4 = 0$$
$$\Rightarrow \sqrt{2}(1-\omega^2) = \pm\omega^2 \Rightarrow \sqrt{2} = (\sqrt{2} \pm 1)\omega^2$$

$\Rightarrow \quad \omega^2 = 2 \pm \sqrt{2}$

~~let x(t)~~

For $\omega^2 = 2+\sqrt{2}$,

let $x(t) = A e^{i\omega t}$

$\phi(t) = B e^{i\omega t}$

Plugging these into the ODE's, we get

$$-2A\omega^2 \cancel{B} - B\omega^2 = -2A$$

$$+\omega^2 A + \omega^2 B = +B$$

$$\Rightarrow \quad A = \left(\frac{1-\omega^2}{\omega^2}\right) B$$

$$\Rightarrow \quad A = \left(\frac{1-2-\sqrt{2}}{2+\sqrt{2}}\right) B = \frac{-1-\sqrt{2}}{\sqrt{2}(1+\sqrt{2})} B$$

$$\Rightarrow \quad A = -\frac{B}{\sqrt{2}}$$

$$\Rightarrow \quad x(t) = A e^{i\omega t}$$

$$\phi(t) = \sqrt{2}A\, e^{i(\omega t + \pi)}$$

$\Rightarrow$ Amplitude of the pendulum is $\sqrt{2} \times$ amplitude of the ~~mass~~ cart & they are out of phase

For $\omega^2 = 2-\sqrt{2}$,

let $x(t) = C e^{i\omega t}$, $\phi(t) = D e^{i\omega t}$

Plugging these into the equations,

$$A = \left(\frac{1-\omega^2}{\omega^2}\right) B = \left(\frac{1-2+\sqrt{2}}{2-\sqrt{2}}\right) B = \frac{\sqrt{2}-1}{\sqrt{2}(\sqrt{2}-1)} = B/\sqrt{2}$$

$\Rightarrow \quad x(t) = A e^{i\omega t}, \quad \phi(t) = \sqrt{2}A\, e^{i\omega t}$

So, the amplitude of the pendulum = $\sqrt{2} \times$ amplitude of the cart but the two oscillate in phase.