PHYS 100B (Prof. Congjun Wu)
Solution to HW 6

March 4, 2011

Problem 1 (Griffiths 8.2)

(Griffiths 7.31) A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor.

(a) Find the electric and magnetic field in the gap, as functions of the distance s from the axis and the time t. (Assume the charge is zero at $t = 0$)

Solution:

$$\mathbf{E} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}, \sigma = \frac{Q(t)}{\pi a^2}, Q(t) = It \Rightarrow \mathbf{E} = \frac{It}{\varepsilon_0 \pi a^2}\hat{\mathbf{z}}.$$

The displacement current density is $\mathbf{J}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{\mathbf{I}}{A} = \frac{I}{\pi a^2}\hat{\mathbf{z}}$. Drawing an Amperian loop at radius s,

$$\oint \mathbf{B} \cdot d\mathbf{l} = B 2\pi s = \mu_0 I_{d_{enc}} = \frac{\mu_0 I}{\pi a^2}\pi s^2 = \frac{\mu_0 I s^2}{a^2} \Rightarrow \mathbf{B}(s,t) = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}.$$

(b) Find the energy density u_{em} and Poynting vector $\mathbf{S}$ in the gap. Note especially the direction of $\mathbf{S}$. Check that Eq. 8.14

$$\frac{\partial}{\partial t}(u_{mech} + u_{em}) = -\nabla \cdot \mathbf{S}$$

is satisfied.

Solution:

$$
\begin{aligned}
u_{em} &= \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \frac{1}{2}\left(\varepsilon_0\left(\frac{It}{\varepsilon_0 \pi a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right) = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].\\
\mathbf{S} &= \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0}\left(\frac{It}{\varepsilon_0 \pi a^2}\hat{\mathbf{z}}\right) \times \left(\frac{\mu_0 I s}{2\pi a^2}\hat{\phi}\right) = -\frac{I^2 t s}{2\pi^2 \varepsilon_0 a^4}\hat{\mathbf{s}}.\\
\frac{\partial u_{em}}{\partial t} &= \frac{\mu_0 I^2}{\pi^2 a^4}c^2 t = \frac{I^2 t}{\varepsilon_0 \pi^2 a^4}\\
-\nabla \cdot \mathbf{S} &= \frac{I^2 t}{2\pi^2 \varepsilon_0 a^4}\nabla \cdot (s\hat{\mathbf{s}}) = \frac{\partial u_{em}}{\partial t}.
\end{aligned}
$$

(c) Determine the total energy in the gap, as a function of time. Calculate the total power flowing into the gap, by integrating the Poynting vector over the appropriate surface. Check that the power input is equal to the rate of increase of energy in the gap. (Eq. 8.9

$$\frac{dW}{dt} = -\frac{d}{dt}\int_V \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_S (\mathbf{E} \times \mathbf{B}) \cdot d\mathbf{a}$$

- in the case $W = 0$, because there is no charge in the gap). [If you are worried about the fringing fields, do it for a volume of radius $b < a$ well inside the gap.]

Solution:

$$U_{em} = \int u_{em} w 2\pi s \, ds = 2\pi w \int_0^b \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right] s \, ds = \frac{\mu_0 w I^2 b^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right].$$
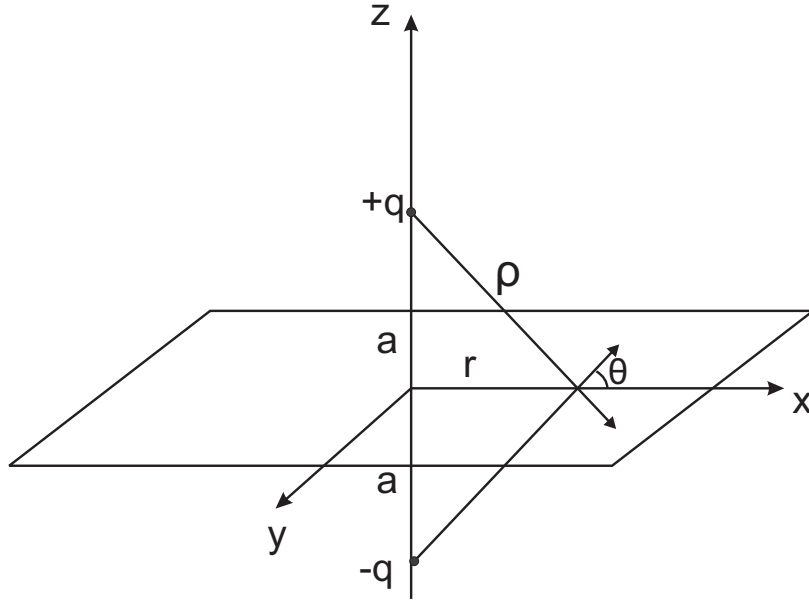
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Figure 1: Problem 8.4

Over a surface at radius b,

$$P_{in} = -\int \mathbf{S} \cdot d\mathbf{a} = \frac{I^2 tb}{2\pi^2 \varepsilon_0 a^4} \hat{\mathbf{s}} \cdot (2\pi bw\hat{\mathbf{s}}) = \frac{I^2 twb^2}{\pi \varepsilon_0 a^4}.$$

$$\frac{dU_{em}}{dt} = \frac{\mu_0 w I^2 b^2}{\pi a^4} c^2 t = P_{in}.$$

Problem 2 (Griffiths 8.4)

(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

Solution:

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z.$$

For the xy plane $da_x = da_y = 0$, $da_z = -rdrd\phi$ (I'll calculate the force on the upper charge).

$$(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z = \epsilon_0 \left(E_z E_z - \frac{1}{2} E^2 \right) (-rdrd\phi).$$

Now $\mathbf{E} = \frac{1}{4\pi\epsilon_0} 2 \frac{q}{\rho^2} \cos\theta \hat{\mathbf{r}}$, and $\cos\theta = \frac{r}{\rho}$, $\rho = \sqrt{r^2 + a^2}$, as shown in Fig.(1), so $E_z = 0$,

$$E^2 = \left(\frac{q}{2\pi\epsilon_0} \right)^2 \frac{r^2}{(r^2 + a^2)^3}.$$

Therefore

$$\begin{aligned} F_z &= \frac{1}{2}\epsilon_0 \left(\frac{q}{2\pi\epsilon_0} \right)^2 2\pi \int_0^\infty \frac{r^3 dr}{(r^2 + a^2)^3} = \frac{q^2}{4\pi\epsilon_0} \frac{1}{2} \int_0^\infty \frac{u\,du}{(u + a^2)^3} \\ &= \frac{q^2}{4\pi\epsilon_0} \frac{1}{2} \left[-\frac{1}{(u + a^2)} + \frac{a^2}{2(u + a^2)^3} \right] \Big|_0^\infty = \frac{q^2}{4\pi\epsilon_0} \frac{1}{(2a)^2}. \end{aligned}$$

(b) Do the same for charges that are opposite in sign.

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Solution: Now $\mathbf{E} = -\frac{1}{4\pi\epsilon_0} 2\frac{q}{\rho^2}\sin\theta\hat{\mathbf{z}}$, and $\sin\theta = \frac{a}{\rho}$,

$$E^2 = E_z^2 = \left(\frac{qa}{2\pi\epsilon_0}\right)^2 \frac{1}{(r^2+a^2)^3}.$$

$$(\overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a})_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2 \frac{rdrd\phi}{(r^2+a^2)^3}.$$

Therefore

$$F_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2 2\pi \int_0^\infty \frac{rdr}{(r^2+a^2)^3} = -\frac{q^2a^2}{4\pi\epsilon_0}\left[-\frac{1}{4}\frac{1}{(r^2+a^2)^2}\right]\bigg|_0^\infty = -\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}.$$

Problem 3 (Griffiths 8.5)

Consider an infinite parallel plate capacitor, with the lower plate (at $z=-d/2$) carrying the charge density $-\sigma$, and the upper plate (at $z=d/2$) carrying the charge density σ.

(a) Determine all nine elements of the stress tensor, in the region between the plates. Display your answer as a 3×3 matrix:

$$\overleftrightarrow{\mathbf{T}} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

Solution:

$$E_x = E_y = 0, E_z = -\frac{\sigma}{\epsilon_0}.$$

$$T_{xx} = T_{yy} = -\frac{\epsilon_0}{2}E^2 = -\frac{\sigma^2}{2\epsilon_0}; T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E^2 = \frac{\sigma^2}{2\epsilon_0}.$$

$$\overleftrightarrow{\mathbf{T}} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Use Eq. 8.22 to determine the force per unit area on the top plate. Compare Eq. 2.51.

Solution:

$$\mathbf{F} = \int \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}$$

($\mathbf{S}=0$ since $\mathbf{B}=0$). Integrate over the xy plane: $d\mathbf{a} = -dxdy\hat{\mathbf{z}}$, (negative because outward with respect to a surface enclosing the upper plate). Therefore

$$F_z = \int T_{zz}da_z = -\frac{\sigma^2}{2\epsilon_0}A.$$

The force per unit area on the top plate is

$$\mathbf{f} = \frac{\mathbf{F}}{A} = -\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{z}}.$$

(c) What is the momentum per unit area, per unit time, crossing the xy plane (or any other plane parallel to that one, between the plates)?

Solution:

$$-T_{zz} = \frac{\sigma^2}{2\epsilon_0}$$

is the momentum per unit area, per unit time, crossing the xy plane perpendicular to z.

(d) At the plates this momentum is absorbed, and the plates recoil (unless there is some nonelectrical force holding them in position). Find the recoil force per unit area on the top plate, and compute your answer to (b) [Note: This is not an additional force, but rather an alternative way of calculating the same force - in (b) we got it from the force law, and in (d) we did it by conservation of momentum].

Solution: The recoil force is the momentum delivered per unit time, so the force per unit area on the top plate is

$$\mathbf{f} = -\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{z}}.$$

(same as (b)).

Problem 4 (Griffiths 8.7)

In Ex. 8.4, suppose that instead of turning off the magnetic field (by reducing I) we turn off the electric field, by connecting a weakly conducting radial spoke between the cylinders. (We'll have to cut a slot in the solenoid, so the cylinders can still rotate freely.) From the magnetic force on the current in the spoke, determine the total angular momentum delivered to the cylinders, as they discharge (they are now rigidly connected, so they rotate together). Compare the initial angular momentum stored in the fields (Eq. 8.35). (Notice that the mechanism by which angular momentum is transferred from the fields to the cylinders is entirely different in the two cases: in Ex. 8.4 it was Faraday's law, but here it is the Lorentz force law.)

Solution:

$$\begin{aligned}
\mathbf{B} &= \mu_0 n I \hat{\mathbf{z}}, a < r < R; \\
\mathbf{B} &= 0, r > R.
\end{aligned}$$

The force on a segment dr of spoke is

$$d\mathbf{F} = I' d\mathbf{l} \times \mathbf{B} = I' \left(dr\hat{\mathbf{r}}\right) \times \left(\mu_0 n I \hat{\mathbf{z}}\right) = -I'\mu_0 n I dr\hat{\phi}.$$

The torque on the spoke is

$$\mathbf{N} = \int \mathbf{r} \times d\mathbf{F} = I'\mu_0 n I \int_a^R r dr \left(-\hat{\mathbf{r}} \times \hat{\phi}\right) = I'\mu_0 n I \frac{1}{2} \left(R^2 - a^2\right)\left(-\hat{\mathbf{z}}\right).$$

Therefore the angular momentum of the cylinders is

$$\begin{aligned}
\mathbf{L} &= \int \mathbf{N} dt = -\frac{1}{2}\mu_0 n I \left(R^2 - a^2\right) \hat{\mathbf{z}} \int I' dt \\
&= -\frac{1}{2}\mu_0 n I Q \left(R^2 - a^2\right) \hat{\mathbf{z}}.
\end{aligned}$$

(in agreement with Eq. 8.35).