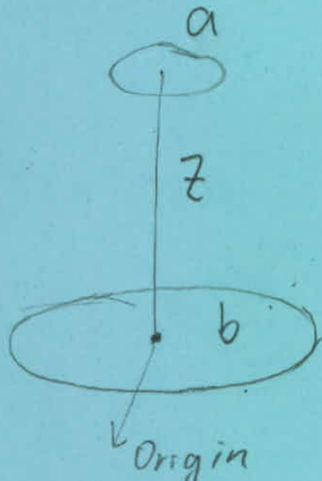
Problem 7.20

(a) The field above the center of a circular loop is

$$\vec{B} = \frac{\mu_0 I}{2} \frac{b^2 \hat{z}}{(b^2+z^2)^{\frac{3}{2}}}$$



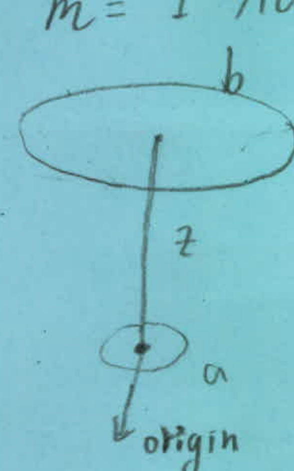
$$\Phi = \int \vec{B} \cdot d\vec{a}$$

$$= B \cdot \pi a^2$$

$$= \frac{\mu_0 \pi I a^2 b^2}{2(b^2+z^2)^{\frac{3}{2}}}$$

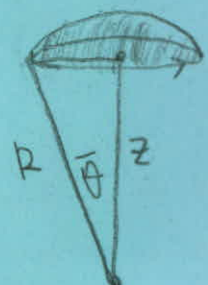
(b) The field of a magnetic dipole $\vec{m}$ is

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{m}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}), \quad \text{where } m = I \cdot \pi a^2$$



$$\Phi = \int \vec{B} \cdot d\vec{a}$$

Consider a partial sphere bound by the big loop and centered at origin.



The radius of the sphere is

$$R = \sqrt{z^2+b^2}$$

$$\bar{\theta} = \tan^{-1}\left(\frac{b}{z}\right)$$

For this parted sphere, we can write $d\vec{a}$ as

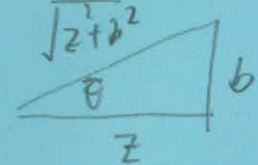
$(R\,d\theta \cdot R\sin\theta\,d\phi)\,\hat{r}$. Thus. $\vec{B}\cdot d\vec{a}$

$$= \frac{\mu_0}{4\pi}\frac{m}{R^3}\cdot(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})\cdot(R^2\sin\theta\,d\theta\,d\phi\,\hat{r})$$

$$= \frac{\mu_0}{2\pi}\frac{m}{R}\cos\theta\sin\theta\,d\phi\,d\theta$$

We integrate over the partial sphere.

$$\int_A \vec{B}\cdot d\vec{a} = \int_0^{2\pi}\int_0^{\bar{\theta}} \frac{\mu_0}{2\pi}\frac{m}{R}\cos\theta\sin\theta\,d\theta\,d\phi$$



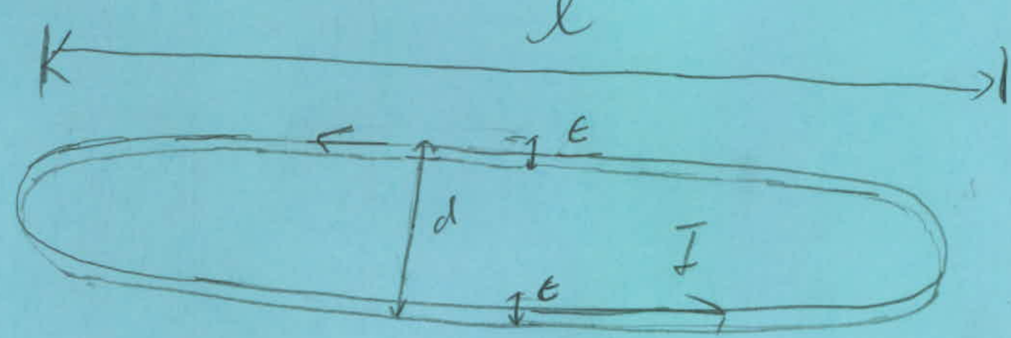
$$= \frac{\mu_0 m}{R}\int_0^{\bar{\theta}} \sin\theta\,d(\sin\theta) = \frac{\mu_0 m}{R}\frac{\sin^2\bar{\theta}}{2}$$

$$= \frac{\mu_0 m}{R}\cdot\frac{1}{2}\times\frac{b^2}{z^2+b^2} = \boxed{\frac{\mu_0 I\pi a^2}{2}\times\frac{b^2}{(z^2+b^2)^{\frac{3}{2}}}}$$ Same as (a)

(c) Since $\Phi_1 = \Phi_2$

, $M_{12} = M_{21} = \frac{\Phi}{I} = \boxed{\frac{\mu_0 I\pi a^2 b^2}{2(z^2+b^2)^{\frac{3}{2}}}}$

Problem 7.23



The field of a longe straight line is

$$B = \frac{\mu_0}{2\pi}\frac{I}{s}$$

Consider flux from one side of the loop.

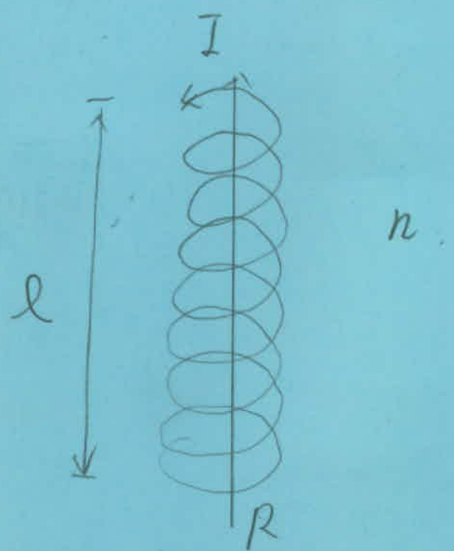
$$\Phi = \int \vec{B}\cdot d\vec{a} = \int_{\epsilon}^{d-\epsilon}\left(\frac{\mu_0 I}{2\pi s}\right) l\, ds$$

$$= \frac{\mu_0 I l}{2\pi}\ln\frac{d-\epsilon}{\epsilon}$$

The total flux is twice as Φ

$$\Phi_{tot} = \frac{\mu_0 I l}{\pi}\ln\frac{d-\epsilon}{\epsilon}$$

Problem 7.26.



(a)

$B = \mu_0 n I$

The flux through the solenoid is

$$\Phi = (\mu_0 n I)\cdot(\pi R^2)\cdot\underbrace{(l n)}_{\#\text{ of loops}}$$

$$\Rightarrow L = \frac{\Phi}{I} = \mu_0 n^2 R^2 \pi l$$

$$W = \frac{1}{2} L I^2 = \frac{1}{2}\mu_0 \pi n^2 R^2 I^2 l$$

(b) $W = \frac{1}{2}\oint \vec{A}\cdot\vec{I}\, dl$, dl: segment of the wire

$\vec{A} = \left(\frac{\mu_0 n I}{2}\right)\cdot R\hat{\phi}$ (Eq. 5.70)

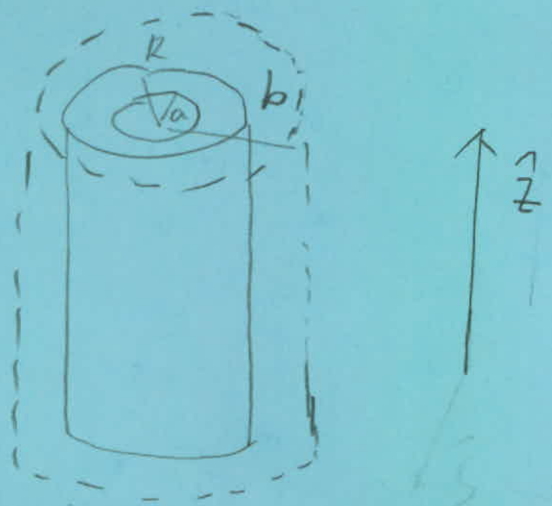
$\vec{I} = I\hat{\phi}$

$$\Rightarrow W = \frac{\mu_0 n I^2 R}{4}\int dl = \frac{\mu_0 n I^2 R}{4}\cdot\overbrace{\left[(l\cdot n)\cdot 2\pi R\right]}^{\text{total length}} = \frac{1}{2}\mu_0 \pi n^2 R^2 I^2 l.$$

(c)

$$W = \frac{1}{2\mu_0}\int_{\text{All Space}} B^2 d\tau = \frac{\mu_0^2 n^2 I^2}{2\mu_0}\int_{\text{Cylinder}} d\tau = \frac{\mu_0 n^2 I^2}{2}\pi R^2 l$$

(d) $W=\frac{1}{2\mu_0}\left[\int B^2\,d\tau - \oint (\vec{A}\times\vec{B})\cdot d\vec{a}\right]$ — ①



$\int B^2\,d\tau$

$=\mu_0^2 n^2 I^2 \int d\tau$

$=\mu_0^2 n_0^2 I^2 \cdot \pi(R^2-a^2)\cdot l$ — ②

$\oint (\vec{A}\times\vec{B})\cdot d\vec{a}$

$=\underbrace{\int (\vec{A}\times\vec{B})\cdot d\vec{a}}_{\text{inner surface}} + \underbrace{\int (\vec{A}\times\vec{B})\cdot d\vec{a}}_{\text{outer surface}}$

$d\vec{a}=[a\,d\phi\,dz\,(-\hat{s})]$

$=\int\left[\left(\frac{\mu_0 n I}{2}\cdot a\hat{\phi}\right)\times(\mu_0 n I\,\hat{z})\right]\cdot d\vec{a} + 0$

$=-\mu_0^2 n^2 a^2 \pi l$ — ②

Plug ② and ③ in ①.

$\Rightarrow W=\frac{1}{2}\mu n^2 I^2 R^2 \pi l$