PHYS 100B (Prof. Congjun Wu)
Solution to HW 2

January 8, 2011

Problem 1 (Griffiths 5.8)

(a) Find the magnetic field at the center of a square loop, which carries a steady current I. Let R be the distance from center to side (Fig. ?).

Solution: ($B = \sqrt{2}\mu_0 I/(\pi R)$.)

By using the Biot-Savart law, we can calculate the contribution from each side of the square wire separately. We find each side contributes the same magnetic field at the center of the square. By using Eq. 5.35 (please refer to Example 5.5), with $s = R$, $\theta_2 = -\theta_1 = \pi/4$, we get

$$B = 4 \times \frac{\mu_0 I}{4\pi R}(\sin\theta_2 - \sin\theta_1) = 4 \times \frac{\mu_0 I}{4\pi R}\left(\sin\frac{\pi}{4} - \sin(-\frac{\pi}{4})\right) = 4 \times \frac{\mu_0 I}{4\pi R}(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) = \sqrt{2}\frac{\mu_0 I}{\pi R}.$$

(b) Find the field at the center of a regular n-sided polygon, carrying a steady current I. Again, let R be the distance from the center to any side.

Solution: ($B = n\mu_0 I/(2\pi R) \times \sin(\pi/n)$.)

Similar to the above analysis, we now have n sides. By using Eq. 5.35, with $s = R$, $\theta_2 = -\theta_1 = \frac{1}{2} \cdot 2\pi/n = \pi/n$, we get

$$B = n \times \frac{\mu_0 I}{4\pi R}(\sin\theta_2 - \sin\theta_1) = n \times \frac{\mu_0 I}{4\pi R}\left(\sin\frac{\pi}{n} - \sin(-\frac{\pi}{n})\right) = n \times \frac{\mu_0 I}{2\pi R}\sin\frac{\pi}{n}.$$

(c) Check that your formula reduces to the field at the center of a circular loop, in the limit $n \to \infty$.

Solution: ($B = \mu_0 I/(2R)$.)

At the limit $n \to \infty$, $\pi/n \ll 1$,

$$\sin\frac{\pi}{n} \approx \frac{\pi}{n}.$$

$$B \approx n \times \frac{\mu_0 I}{2\pi R} \times \frac{\pi}{n} = \frac{\mu_0 I}{2R}.$$

Problem 2 (Griffiths 5.13)

A steady current I flows down a long cylindrical wire of radius a (Fig. ?). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is diastributed in such a way that I is proportional to s, the distance from the axis.

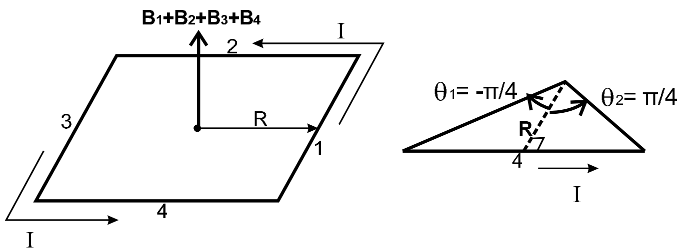

Figure 1: Problem 5.8

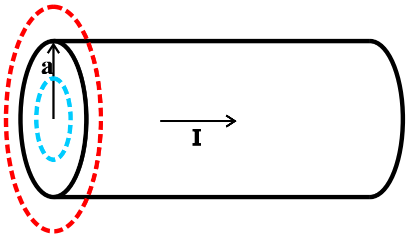

Figure 2: Problem 5.13

Solution: (a) $\mathbf{B} = 0, r < a; \mathbf{B} = \mu_0 I/(2\pi r)\hat{\phi}, r > a$.(b) $\mathbf{B} = \mu_0 I r^2/(2\pi a^3)\hat{\phi}, r < a; \mathbf{B} = \mu_0 I/(2\pi r)\hat{\phi}, r > a$.
By Ampère's law, we have

$$\int_{loop} \mathbf{B} \cdot d\mathbf{l} = B 2\pi r = \mu_0 I_{encircled}.$$

Here, r is the radius of the Amperian loop. Noticing the rotational symmetry with respect the axis of the cylinder, we take the Amperian loop, as shown in the dashed line in the figure, so that the magnitude of $\mathbf{B}$ is constant around this loop when we do the integration above.

The dashed blue loop is designed for finding the magnetic field inside the cylindrical wire; the dashed red loop is for detecting the magnetic field outside.

For case (a), the current is distributed over the outside surface of the cylinder of radius a.
For $r < a$, the blue loop does not encircle any current, $I_{encircled} = 0$. $\Rightarrow B = 0$, $\mathbf{B} = 0$.
For $r > a$, the red loop encircles all the current, $I_{encircled} = I$. $\Rightarrow B = \mu_0 I/(2\pi r)$.

For case (b), $J \propto r$. Let

$$J = kr.$$

The current

$$I = \int J dA,$$

where, J is the volume current density, $dA = (2\pi r')\, dr'$ is the area element between the circles of radius r' and of radius $(r' + dr')$.

$$I = \int_0^a (kr')\,(2\pi r')\, dr' = \frac{2\pi k a^3}{3}. \Rightarrow k = \frac{3I}{2\pi a^3}.$$

For $r < a$,

$$\begin{aligned} I_{encircled} &= \int_0^r (kr')\,(2\pi r')\, dr' = \frac{2\pi k r^3}{3} = I\frac{r^3}{a^3}. \\ B &= \frac{\mu_0 I_{encircled}}{2\pi r} = \frac{\mu_0 I r^2}{2\pi a^3}. \end{aligned}$$

For $r > a$,

$$\begin{aligned} I_{encircled} &= I. \\ B &= \frac{\mu_0 I_{encircled}}{2\pi r} = \frac{\mu_0 I}{2\pi r}. \end{aligned}$$

Problem 3 (Griffiths 5.16)

A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v.

(a) Find the magnetic field between the plates and also above and below them.

Solution: $\left(B = \begin{cases} \mu_0 \sigma v, between\, the\, plates; \\ 0, elsewhere. \end{cases}\right)$

Please refer to example 5.8, where the surface current density $\mathbf{K} = \sigma v \mathbf{x}$ for the upper plate, $\mathbf{K} = -\sigma v \mathbf{x}$ for the lower plate. Using the result from example 5.8, the upper plate produces the magnetic field

$$\mathbf{B}_{upper} = \begin{cases} +\frac{\mu_0}{2}\sigma v \mathbf{y}, below\, the\, upper\, plate; \\ -\frac{\mu_0}{2}\sigma v \mathbf{y}, above\, the\, upper\, plate. \end{cases}$$

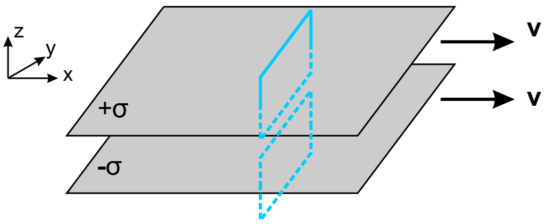

Figure 3: Problem 5.16

The lower plate produces the magnetic field

$$\mathbf{B}_{lower} = \begin{cases} -\frac{\mu_0}{2}\sigma v\mathbf{y}, below\,the\,lower\,plate; \\ +\frac{\mu_0}{2}\sigma v\mathbf{y}, above\,the\,lower\,plate. \end{cases}$$

$\Rightarrow$

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_{upper} + \mathbf{B}_{lower}. \\ B &= \begin{cases} \mu_0\sigma v, between\,the\,plates; \\ 0, elsewhere. \end{cases} \end{aligned}$$

(b) Find the magnetic force per unit area on the upper plate, including its direction.

Solution: ($f_m = \mu_0\sigma^2 v^2/2\hat{\mathbf{z}}$)

Since the upper plate cannot feel the magnetic field porduced by itself, the magnetic force here corresponds to the magnetic field produced by the lower plate and felt by the upper plate.

Lorentz force law says $\mathbf{F} = \int (\mathbf{K} \times \mathbf{B})\, d\mathbf{a}$, so the force per unit area is

$$\mathbf{f}_m = \mathbf{K} \times \mathbf{B}.$$

Here, $\mathbf{K} = \sigma v\hat{\mathbf{x}}$ for the upper plate, $\mathbf{B} = \mathbf{B}_{lower} = \frac{\mu_0}{2}\sigma v\hat{\mathbf{y}}$.

$$\mathbf{f}_m = \frac{\mu_0}{2}\sigma^2 v^2\hat{\mathbf{z}}.$$

(c) At what speed v would the magnetic force balance the electrical force?

Solution: ($v = c$, the speed of light**).**

The electric field of the lower plate is $\sigma/(2\varepsilon_0)$. The attractive electric force per unit area on the upper plate is

$$\mathbf{f}_e = -\frac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}}.$$

They balance if

$$\frac{\mu_0}{2}\sigma^2 v^2 = \frac{\sigma^2}{2\varepsilon_0} \Rightarrow v = \frac{1}{\sqrt{\varepsilon_0\mu_0}} = c.$$

Problem 4 (Griffiths 5.17)

Show that the magnetic field of an infinite solenoid runs parallel to the axis, regardless of the cross-sectional shape of the coil, as long as that shape is constant along the length of the solenoid. What is the magnitude of the field, inside and outside of such a coil? Show that the toroid field (5.58) reduces to the solenoid field, when the radius of the donut is so large that a segment can be considered essentially straight.

Proof:

(Example 5.9 and Example 5.10 give us the spirit for solving this problem.)

Let us first pick up a point $M\,(0, y, 0)$ located on the y axis, and then use the Biot-Savart law to calculate the magnetic field at this point.

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\mathbf{l} \times \mathbf{r}}{r^3}.$$

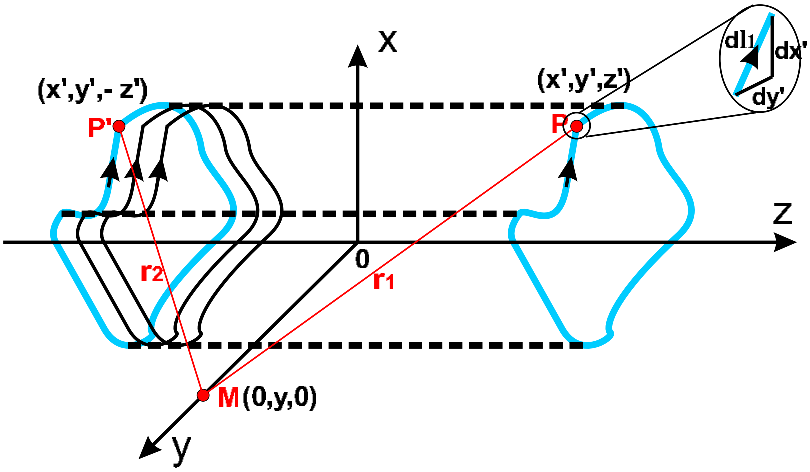

Figure 4: Problem 5.17

To fully use the translational symmetry along the z-axis, we pick up line elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$ at points P(x', y', z') and P'$(x', y', -z')$ symmetrically with respect to the x-y plane, and consider their contribution to the integration element $d\mathbf{B}$ together:

$$
\begin{aligned}
d\mathbf{B} &= \frac{\mu_0 I}{4\pi}\left(\frac{d\mathbf{l}_1 \times \mathbf{r}_1}{r_1^3} + \frac{d\mathbf{l}_2 \times \mathbf{r}_2}{r_2^3}\right), \\
\mathbf{r}_1 &= \mathbf{r}_M - \mathbf{r}_P = -x'\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} - z'\hat{\mathbf{z}}, \\
\mathbf{r}_2 &= \mathbf{r}_M - \mathbf{r}_{P'} = -x'\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + z'\hat{\mathbf{z}}, \\
r_1 &= r_2 = \sqrt{x'^2 + (y - y')^2 + z'^2} \equiv \tilde{r}, \\
d\mathbf{l}_1 &= d\mathbf{l}_2 = dx'\hat{\mathbf{x}} + dy'\hat{\mathbf{y}} \equiv d\tilde{\mathbf{l}}.
\end{aligned}
$$

$\Rightarrow d\mathbf{B} = \mu_0 I/(4\pi)\,\frac{d\tilde{\mathbf{l}} \times (\mathbf{r}_1 + \mathbf{r}_2)}{\tilde{r}^3}$. Since $d\tilde{\mathbf{l}}$ and $(\mathbf{r}_1 + \mathbf{r}_2)$ are in the same x-y plane, $d\mathbf{B} \parallel d\tilde{\mathbf{l}} \times (\mathbf{r}_1 + \mathbf{r}_2)$ is always along the z-axis, which is perpendicular to the x-y plane.

$$
\begin{aligned}
d\mathbf{B} &= \frac{\mu_0 I}{4\pi}\frac{d\tilde{\mathbf{l}} \times (\mathbf{r}_1 + \mathbf{r}_2)}{\tilde{r}^3} = \frac{\mu_0 I}{4\pi}\frac{(dx'\hat{\mathbf{x}} + dy'\hat{\mathbf{y}}) \times (-2x'\hat{\mathbf{x}} + 2\,(y - y')\,\hat{\mathbf{y}})}{\left(\sqrt{x'^2 + (y - y')^2 + z'^2}\right)^3} \\
&= \frac{\mu_0 I}{4\pi}\frac{(2\,(y - y')\,dx' + 2x'dy')}{\left(\sqrt{x'^2 + (y - y')^2 + z'^2}\right)^3}\hat{\mathbf{z}}.
\end{aligned}
$$

Using Ampère's law, we find the magnetic field is

$$
B = \begin{cases} \mu_0 n I, \textit{inside the coil}; \\ 0, \textit{outside the coil}. \end{cases}
$$

where n is the number of turns in a unit length, which equals the total number of turns, N, divided by the length of the circumference, $2\pi s$, for a toroid with large radius s.